Creation of Master curves without adopting a Rheological Model

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ABSTRACT

The creation of master curves is described by using ordinarily mathematical functions in a fitting process. This is done in an Excel program with the aid of the Solver option. A power function is suited for describing the stiffness master curve. For the modeling of the phase lag master curve a power function in combination with a Gauss curve is more appropriate. For the investigated mixes the limit values for the stiffness master curves are realistic in contrast with the predicted model values for the phase lags at \( f = 0 \) Hz. This is due to the appearance of a local maximum between 0.1 and 2 Hz of the ‘normalized’ frequency and the lack of data for frequencies below 0.1 Hz. For the development of the model measurements have been used of six different Porous Asphalt mixes (ZOAB).
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1. Introduction

It is well known that for non-modified bitumen and mixes, and for some content also for modified bitumen, a temperature $\Leftrightarrow$ frequency exchange relationship exists. This means that the same complex stiffness modulus value measured at temperature $T_1$ and frequency $f_1$ can also be found at temperature $T_2$ and frequency $f_2$. The loading conditions temperature & frequency are exchangeable. For a visualization of this behaviour it is common use to:

1) Choose a reference temperature $T_r$
2) Multiply the frequencies carried out at a temperature $T_i$ with a constant factor $X_i$ ($X_i = 1$ for $T_i = T_r$).
3) Plot the modulus and the phase lag of the measured complex stiffness moduli as a function of the multiplied frequencies.
4) Vary the multiplication factor $X_i$ per temperature (except for $T_r$) in such away that the measured values (modulus and phase lag) follow a smooth curve
5) The obtained smooth curves are known as “Master curves”
6) Plot the multiplication factors $X_i$ as a function of the temperature. If the logarithm is taken of $X_i$ for the plot value, these values should often lie on a straight curve.

It is obvious that the described procedure is in fact a try & error solution procedure. Therefore often a rheological model is chosen to simulate the behaviour of the material as a function of the frequency. Some properties in the model are supposed to depend on the temperature. With this approach it is more convenient to plot the imaginary stiffness value (loss modulus) as a function of the real stiffness value (storage modulus). If a temperature – frequency exchangeability exists the measured values should lie already on a smooth curve. The chosen rheological model creates the possibility to determine this curve by eliminating the frequency from the equations for the loss and storage modulus. In this way it is possible to determine by curve fitting for each temperature the best fitting values for the properties, which are assumed to depend on the temperature.

In this report a procedure is described how to create with a simple optimisation program a mastercurve and determine the temperature – frequency exchangeability without using a rheological model. But it is still necessarily to adopt mathematical functions, which describe the desired
smooth curves, but these equations are very ordinarily and based on observed behaviours.

2. Chosen Curve Forms

2.1. General
If the modulus of the complex modulus values is plotted as a function of the logarithm value for the applied frequency, five findings are quite clear:

- For large frequencies the modulus tends to a limited high value (glass modulus)
- The phase lag will decrease to a small value or even becomes zero (elastic)
- If the frequency goes to zero the modulus will tend to a small constant value or even becomes zero
- The phase lag increases in first instance but often it passes a maximum and decreases afterwards to a lower value; the phase lag will always be smaller than 90°.
- The whole modulus curve has a S-shaped form.

There are several simple equations, which can describe this behaviour:

\[ F_1 = A - \frac{2B}{\pi} \cdot \arctg \left( C \cdot \ln \left( \frac{f}{f_0} \right) \right); \quad f \to 0 \Rightarrow F_1 = A + B; \quad f \to +\infty \Rightarrow F_1 = A - B \quad [1] \]

\[ F_1 = A - B \cdot e^{-\frac{f}{f_0}}; \quad f = 0 \Rightarrow F_1 = A - B; \quad f \to +\infty \Rightarrow F_1 = A \quad [2] \]

\[ F_1 = A - B \cdot \frac{f^C}{f^C + D}; \quad f = 0 \Rightarrow F_1 = A; \quad f \to +\infty \Rightarrow F_1 = A - B \quad [3] \]

\[ F_1 = A \cdot \frac{B \cdot f^C + 1}{D \cdot f^E + 1}; \quad f = 0 \Rightarrow F_1 = A; \quad f \to +\infty \Rightarrow F_1 = 0 \quad \text{if} \quad E > C \quad [4] \]

\[ F_2 = A \cdot e^{-B \left( \ln \left( \frac{f}{f_0} \right) \right)^2}; \quad f = 0 \text{ or } +\infty \Rightarrow F_2 = 0; \quad f = f_0 \Rightarrow F_2 = A \quad [5] \]

\[ F_2 = \frac{A}{1 + B \left( \ln \left( \frac{f}{f_0} \right) \right)^2}; \quad f = 0 \text{ or } +\infty \Rightarrow F_2 = 0; \quad f = f_0 \Rightarrow F_2 = A \quad [6] \]
There are of course more equations but the equations above are very simple and easy to use for a regression. The function $F_1$ describes a S-shape while function $F_2$ is a symmetrical curve round $f=f_0$ and goes to 0 for $f = 0$ and $f \to +\infty$.

Examples of the curves for equations 1, 2, 3, 5 and 6 are given in the figures 1 to 6 for different values of the parameters. Later on the curve for equation 4 will be discussed in paragraph 2.4.
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Figure 2. Shape of Exponential function $F_1$ [eq. 2]
Figure 3. Shape of Power function $F_1$ [eq. 3]

Figure 4. Shape of Gauss function $F_2$ [eq. 5]
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In view of common measured figures for the stiffness modulus as a function of the frequency the shape of the power function \( F_1 \) [eq. 3] is quite acceptable for a regression. In case of the data measured for the phase lag a function \( F_2 \) has to be added. Both functions \( F_2 \) are acceptable. For the regression program the Gauss function \( F_2 \) [eq. 5] has been chosen.

However, some remarks have to be made on the regression procedure itself. It is advised to carry out the regression in two steps. The first step is to use the power function \( F_1 \) [eq. 3] for a regression on the stiffness modulus alone and to use shift factors for the frequencies (for those tests which are not carried out at the chosen reference temperature). This will already lead to acceptable master curves for both the stiffness modulus and in many cases also for the phase lag. In the second step a regression can be performed on stiffness modulus and phase lag together. Due to the random errors and noise it is advised to formulate the regression formula for the phase lag as follows:

\[
\varphi = \varphi_0 \cdot F_1 + \varphi_1 \cdot F_2 \quad ; \quad \varphi_{\text{Minimum}} \leq \varphi_0 \leq \varphi_{\text{Maximum}}
\]  

[7]

The two boundary values for the phase lag \( \varphi_0 \) are based on the expected error in the measured phase lag and the values measured at the lowest frequencies and highest temperatures. An example is given in Annex A.
This procedure is not satisfactorily and hampers an objective approximation of the phase lag at f=0 Hz. Therefore equation [4] is also applied and will be discussed in paragraph 2.4.
2.2. Modulus equation

Based on experience the following equation will describe the modulus behaviour very well.

\[
S_{\text{mod}} = \text{Modulus at frequency } f ; \\
S_0 = \text{Modulus at frequency } f = 0 ; \\
S_\infty = \text{Modulus at frequency } f \rightarrow \infty ; \\
f = \text{applied frequentie [Hz]} ; \\
C, D = \text{Constants, determining the S-shape}
\]

\[
S_{\text{mod}} = S_0 + (S_\infty - S_0) \frac{f^C}{f^C + D}
\]

2.3 Phase lag equation

The basic form for the master curve of the phase lag is also very well described by a power function [eq. 1]. Because sometimes a local maximum in the phase lag is present for \( f > 0 \) Hz a second function can be added in order to obtain a better fit.

\[
F_1 + F_2 = \varphi_0 \cdot \frac{1}{\beta \cdot f^\gamma + 1} + \varphi_1 \cdot e^{-\mu \left( \ln \left( \frac{f}{f_0} \right) \right)^2}
\]

If no restrictions are made with respect to the phase lag at \( f=0 \) Hz it can happen that the regression will lead to a perfect fit in view of the measured values (all for \( f > 0 \) Hz) but an unrealistic (extrapolated) value (\( = \varphi_0 \)) for the phase lag at \( f = 0 \) Hz. An example is given in Annex 1. Due to unwanted necessarily restrictions in the regression process another single equation was developed.

2.4 Modified Phase lag equation

The summation of two functions is not quite satisfactorily if an extrapolated (limit) value has to be obtained. Therefore the applicability of equation [4] is investigated also. This function already can have a local maximum and has only 5 regression parameters in contrast with the 6 regression parameters for equation [9], which is build up out of two functions. The used formulation is given in equation [10]

\[
F_1 = \varphi_0 \cdot \frac{A \cdot f^\alpha + 1}{B \cdot f^\beta + 1} \quad \text{with} \quad \beta > \alpha
\]

The forms of equations [10] and [8] are given in figure 6. Although the applicability should be less equation [10] still resembles very good the measured phase lag curves in practice. In the Excel program (Chapter 3) both solutions for the phase lag curve fitting has been applied. In the first attempt
equation [9] is used leading to excellent fits if the restriction with respect to the phase lag at f=0 Hz was not applied. However, the obtained values for f=0 Hz were not at all realistic. In the second attempt equation [10] is used (after the regression on stiffness in which the shift factors were determined). This approach lead to realistic values at f=0Hz.

**Figure 6. Functions for Stiffness (eq. 8) and Phase lag (eq. 10)**

3. The Excel Program

For the regression analysis the Solver option in the Excel program (Office 200) has been used. Because no rheological model is assumed both the measured modulus and the phase lags of the complex stiffness modulus have been treated as non-related figures. The Solver option is used in a two step approach. First equation [8] is used for the creation of the master curve for the stiffness modulus by optimising the constants $S_0$, $S_\infty$, A and B in equation [8] and applying shift factors for the frequencies, which only depend on the applied temperature. Because a temperature of 20 °C is taken as the reference temperature, the shift factor for 20 °C is by definition 1. In the second step the phase lags, which due to the shift factors for the frequencies already will be on a smooth curve, are used for a regression with the aid of equations [9] or [10]. Using equation [9] it is necessarily to split up the analysis into two steps by performing firstly a regression on the first term in equation [9] and afterwards on both terms. It should be marked that in this approach the shift factors are
only based on a regression for the modulus. A total regression taken both
equation [8] and [9] simultaneously is also possible but lead to negligible
differences in the shift factor equation. For this reason it is advised to use for a
final program equation [10] instead of equation [9].

A last remark can be made on the regression for the shift factor. In this
approach no equation between shift factor and temperature has been adopted
on forehand.

4. Data

The data for the 6 different ZOAB mixes (porous asphalitic mixes), which are
used for this study are given in the (Dutch) report:

“Bepaling van maste curves d.m.v. een dynamisch spectrum (frequentie sweep) in
de vierpuntsbuig-opstelling”; IL-R-04.016; DWW; 2004

(Determination of master curves with the aid of a dynamic spectrum (frequency
sweep) in the four point bending apparatus)

In principle 3 beams per mix were used for the frequency sweep tests. The
tests were performed in controlled deflection mode with a low strain value and
a limited number of cycles per frequency in order to avoid any fatigue damage.
This was checked by applying the start frequency of 0.98 Hz also at the end of
a frequency sweep and to compare both the measured response data. An
overview of the tested beams is given in table 1.

Table 1. Beam codes used in the tests.

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<td>2/3</td>
<td>4/5/6</td>
<td>7/8/9</td>
<td>10/11/12</td>
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<td>4/5/6</td>
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<td>2/3</td>
<td>4/5</td>
<td>7/8/9</td>
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<td>7/8/9</td>
<td>10/11/12</td>
<td>13/14/15</td>
<td>18</td>
</tr>
</tbody>
</table>

In this study all the data has been used in the regression analysis. However, it
is recommended to perform the regression analysis on each beam separately.
Afterwards mean values can be calculated for the regression constants of each mix separately. An approach on all measured values for one mix separately can give convergence problems if the number of tested beams is small (mix F).
5. Results

The results for the regression analysis are given in figures 6 to 29. It should be noted that these figures are obtained from an approach in which equation [9] is used for the regression on the phase lag data with a restriction (range) for the extrapolated phase lag at f=0 Hz.

**Figure 6.** Stiffness master curve Mix A using equation [7]
Figure 7.  Shift factor for creating master curve Mix A at T = 20 °C

![Mastercurve at 20 °C Mix A](image)

Figure 8.  Phase lag master curve Mix A using equation [8]

![Cole-Cole Diagram Mix A](image)
Figure 9.  Cole-Cole diagram Mix A and fitted behaviour.
Figure 10. Stiffness master curve Mix B using equation [7]

Figure 11. Shift factor for creating master curve Mix B at T = 20 °C
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Figure 12. Phase lag master curve Mix B using equation [8]
Figure 13. Cole-Cole diagram Mix B and fitted behaviour.

Figure 14. Stiffness master curve Mix C using equation [7]
Figure 15. Shift factor for creating master curve Mix C at T = 20 °C

Figure 16. Phase lag master curve Mix C using equation [8]

Figure: Mastercurve at 20 °C Mix C
- Phase lag vs. Frequency [Hz]

Figure: Cole-Cole Diagram Mix C
- Storage Modulus [MPa]
- Loss Modulus [MPa]
Figure 17. Cole-Cole diagram Mix C and fitted behaviour.

Figure 18. Stiffness master curve Mix D using equation [7].
Figure 19. Shift factor for creating master curve Mix D at $T = 20^\circ C$
Figure 20. Phase lag master curve Mix D using equation [8]

Figure 21. Cole-Cole diagram Mix D and fitted behaviour.

Figure 22. Mastercurve at 20 °C Mix E
Figure 22. Stiffness master curve Mix E using equation [7]

\[ \text{Shift factor} = 756.04 e^{-0.322 T} \]

\( R^2 = 0.9992 \)

Figure 23. Shift factor for creating master curve Mix E at \( T = 20 \, ^\circ C \)

Mastercurve at 20 \( ^\circ C \) Mix E

Phase lag

\( \text{Measured} \quad \text{Predicted} \)
Figure 24.  Phase lag master curve Mix E using equation [8]

Figure 25.  Cole-Cole diagram Mix E and fitted behaviour.
Figure 26. Stiffness master curve Mix F using equation [7]
Figure 27. Shift factor for creating master curve Mix F at $T = 20^\circ C$

Figure 28. Phase lag master curve Mix F using equation [8]
Figure 29. Cole-Cole diagram Mix F and fitted behaviour.
6. Discussion & Conclusions

Although the three functions [eq. 1,2,3] all show a S-type curve, it turns out that for the mixes at issue the power function [eq.3] yields the best resemblance with the measured stiffness data (see Annex B for the Arctangent function [eq. 1]). In this report only a very simple power function is used with only three parameters. However, the following equation with more parameters can lead even to a better comparison/fit. In view of the possible errors and the already very good fit this approach is not necessarily.

\[ F = \frac{A f^\alpha + B f^\beta + S_0}{C f^\alpha + 1} \; ; \alpha > \beta \; ; B \geq 0 \]

\[ \Rightarrow f \to \infty : F \to \frac{A}{C} = S_\infty \; ; f = 0 : F = S_0 \]

For \( B = 0 \) equation [3] is obtained.

In most cases the phase lag data will also follow a (mirrored) S-type curve but sometimes a local maximum can occur. This implicates that besides the power function (e.g. equation [3]) a second function has to be added for simulating this local maximum. The obtained fit using two independent functions is very good but due to the lack of data at low frequencies and high temperatures the calculated (extrapolated) phase lag at \( f = 0 \) Hz for the reference temperature can turn out to be unrealistic (see Annex A). Therefore it is advised to use a function like equation [10] which already can show alone a local maximum in the phase lag curve.

Thus two conclusions can be drawn:
1. The power function [eq. 3] is very suitable for describing the stiffness master curve and can be used for estimations of the stiffness modulus at \( f = 0 \) Hz and for \( f \to \infty \) Hz.
2. The phase lag master curve can be modelled by the sum of a power function and a “clock” type function like the Gauss curve [eq. 4]. The mathematical description is suited for interpolation but if data are missing for low frequencies the mathematical extrapolation to \( f = 0 \) Hz can lead to unrealistic figures.
3. It is better to use one but more complex power function which allows for a local maximum for the regression on the phase lag data.
It is advised to perform the regression analysis separately. That is to say: In the regression analysis one should use the data obtained for one sample (beam) tested with a frequency sweep at different temperatures. Afterwards mean values can be calculated as representative values for the mix at issue.
Annex A “Combined regression procedure”

Instead of a two or three step approach in the regression procedure (Shift & Stiffness followed by Phase lag) it is of course allowed to carry out a regression in one step by varying all parameters simultaneously. The obtained results and figures for the stiffness master curve and shift factors do not differ much from those obtained in a two step approach. Even the fit through the measured data for the phase lags is much better. From this point of view an combined approach seems to be favourable. But because the fit model for the phase lag is build up out of two functions and no measured value is available for a frequency of 0 Hz the calculated value for f = 0 Hz can be very unrealistic. This is illustrated in figures A.1 to A.4 and table A.1 for mix A using the combined fitting approach with the aid of equations [3] and [8].

**Figure A.1** “Stiffness master curve for Mix A using an combined fit approach”
Figure A.2  “Calculated Shift factor equation for mix A at 20 °C"
**Figure A.3** “Phase lag master curve for mix A (Two step approach)"

![Mastercurve at 20 °C Mix A](image)

**Figure A.4** “Cole-Cole diagram for mix A”

![Cole-Cole Diagram Mix A](image)

**Table A.1** “Regression figures”

<p>| | | | | | |</p>
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<td>$0$</td>
<td>$0$</td>
<td>$\varphi_0$</td>
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<td>First column: Two step approach</td>
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<td>$15248$</td>
<td>$15415$</td>
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<tr>
<td>Second column: Combined approach</td>
<td>$\varphi_1$</td>
<td>$10.04$</td>
<td>$10.72$</td>
<td>$\gamma$</td>
<td>$0.49$</td>
</tr>
<tr>
<td>Note that the phase lag $\varphi_0$ for $f = 0$ Hz drops from $39.9^\circ$ to $32.4^\circ$, which is much too low</td>
<td>$\varphi_1$</td>
<td>$0.343$</td>
<td>$0.339$</td>
<td>$\varphi_1$</td>
<td>$7.37$</td>
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in view of the measured data

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</tr>
<tr>
<td>0.007</td>
<td>9.2</td>
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</table>
Annex B “Arctangent function for the stiffness modulus master curve”

Instead of the power function [3] it is also possible to use the arctangent function [1]. In spite of the 4 constants the form of this function doesn’t fit very well in view of the expected master curve for the stiffness modulus. This is shown in figure B.1 for the results obtained with mix A. Moreover, in the regression it was necessarily to fit on the shift factor, modulus and phase lag in one common step. The calculated shift factors are still on a straight line but as can be seen in figure B.1 the fit between the model and the results is less than with the power function.
**Figure B.1** Stiffness master curve for Mix A at reference temperature of 20 °C using the Arctangent function [1] as a fit model.