THEORY OF THE
FOUR POINT DYNAMIC BENDING TEST

PART I: GENERAL THEORY

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P-DWW-96-008
ISBN-90-3693-712-4
Edition: December 2006

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PREFACE

This document contains the background, theory and interpretation procedures of the

FOUR POINT DYNAMIC BENDING TEST

The 4 point dynamic bending test is used for the investigation of pavement material properties like stiffness modulus and fatigue characteristics. The purpose of this document is to give an overview of the equations which govern the interpretation of the measurements. While the equations for static bending are well known, this is not the case in dynamic bending. In principle the equations\(^1\) are derived for the 4 point dynamic test but the theory can easily be extended to the 3 point dynamic test, which has the same geometrics. The following items are discussed:

Chapter 1: General theory for the bending of a rectangular beam; The derivations of the differential equations are given for a slender beam. It is shown that in case of an asphalt beam only a few terms in the differential equations are important leading to the differential equations for pure bending.

Chapter 2: The static bending case; In this chapter the solution of the basic 4\(^{th}\) order differential equation is derived. It is shown that this solution is equal to the more familiar two (three) equivalent 2\(^{nd}\) order differential equations, which are frequently used in handbooks. Also the influences of the placing of the 4 clamps along the beam are briefly discussed. In the static case it is not relevant that a difference exists between the actual beam length \(L_{\text{tot}}\) and the distance \(L\) between the two outer clamps.

Chapter 3: The dynamic bending of an elastic beam; The exact solution is derived in the case of pure bending. The actual beam length \(L_{\text{tot}}\) is equal to the distance \(L\) between the two outer clamps. Only the mass of the beam is taken into account. The solution consists out of an infinite series of cosine terms. It is shown that a 1\(^{st}\) order approximation (only the first term of the series) covers more than 95 \(\%\) of the exact solution. Using the coefficient of the solution for static bending, a modified approximation will lead to even better results for common asphalt properties. This approximation is mathematically equivalent to the solution for a mass-spring system.

Chapter 4: The dynamic bending of a viscous-elastic material;

\(^1\) For clarity reasons in this 2006 edition the “times” symbol \(x\) is replaced in the equations/formulas by a “dot.”. In the text the product sign \(x\) is still used if necessarily
The exact solution is derived in the case of pure bending. The actual beam length $L_{\text{tot}}$ is equal to the distance $L$ between the two outer clamps. Only the mass of the beam is taken into account. The solution consists out of an infinite series of cosine terms. Again a 1st order approximation covers more than 95% of the exact solution. Because the form of the approximation is equal to the solution for a mass - (viscous-elastic) spring system, an equivalent mass can be given representing the influence of the actual weight of the asphalt beam. Of course this equivalent mass will depend on the place along the beam where the response (deflection) will be measured. Also the solutions for the homogenous differential equation are given, which are needed in the case that the actual beam length $L_{\text{tot}}$ is longer than the distance $L$ between the two outer clamps.

Chapter 5: The influence of (extra) moving masses at the two inner clamps;
In this chapter it is shown how to incorporate these forces in the theory for a viscous-elastic beam, provided that the actual beam length $L_{\text{tot}}$ equals the distance $L$ between the outer clamps. A 1st order approximation is given, which again equals the solution for a mass - (viscous-elastic) spring system. The equivalent mass is given in case of the DWW specifications and if the response is measured in the centre of the beam.

Chapter 6: The influence of the overhanging beam ends ($L_{\text{tot}} > L$) on the deflection for an elastic beam;
For the case that no extra moving masses are present at the two inner clamps, the exact solutions are derived in this chapter for an elastic beam.

Chapter 7: The influence of the overhanging beam ends ($L_{\text{tot}} > L$) on the deflection for a viscous-elastic beam;
For the case that no extra moving masses are present at the two inner clamps, the exact solutions are derived in this chapter for a viscous-elastic beam. It is shown that in case of the DWW specifications ($l_0=450\,\text{mm};\,L=400\,\text{mm}$) the introduced error is very small. An approximation procedure is given if extra moving masses are present.

Chapter 8: Advised interpretation formulas for the back calculation of Stiffness modulus from deflection measurements on asphalt beams; in this chapter an overview is given of the advised equations for the back calculation procedure of measurements carried out in a 4 point dynamic bending test with the DWW and ASTM specifications.
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FOUR POINT BENDING TEST

1. Bending Theory for a Rectangular Beam

1.1 General Theory

The deflections due to shear $V_s$ and due to bending $V_b$ of a rectangular beam, which is vertically loaded along the beam by a force $Q(x,t)$ are governed by two differential equations (assuming homogenous material). The derivations of these two equations are given in many textbooks:

$$\rho \cdot b \cdot h \cdot \frac{\partial^2}{\partial t^2} \left[ V_b(x,t) + V_s(x,t) \right] - \frac{\partial}{\partial x} D = Q(x,t)$$  \hspace{1cm} (1)

$$- \rho \cdot I \cdot \frac{\partial^3}{\partial t^2 \partial x} V_b(x,t) - \frac{\partial}{\partial x} M + D = 0$$  \hspace{1cm} (2)

The moment $M$ is related to the deflection $V_b$ by:

$$M = + E \cdot I \cdot \Phi_b(x,t) = - E \cdot I \cdot \frac{\partial^2}{\partial x^2} V_b(x,t)$$  \hspace{1cm} (3)

The shear force $D$ is given by:

$$D = - \int_{\frac{h}{2}}^{\frac{h}{2}} G \cdot b \cdot \Phi_s(h,x,t)$$  \hspace{1cm} (4)

It is assumed that the function $\Phi_s(h,x,t)$ can be split up into two functions:

$$\Phi_s(h,x,t) = f(h) \cdot \Phi_s(x,t)$$  \hspace{1cm} (5)

Thus $D$ can be related to $V_s$ by:

$$D = - G \cdot \Omega [b,h] \cdot \Phi_s(x,t) = + G \cdot \Omega [b,h] \cdot \frac{\partial}{\partial x} V_s(x,t)$$  \hspace{1cm} (6)

in which the function $\Omega[b,h]$ can be considered as a kind of effective area.

For the given integral of $b \times \int [h] = \Omega [b,h]$ it is often assumed that this value equals $\frac{2}{3} b h$ (two thirds of the original rectangular area). However the exact value is not important in the case of slender beams.

Replacing the moment $M$ and the shear force $D$ by these equations will lead to the following two basic equations for the bending of a rectangular beam.

$$\rho \cdot b \cdot h \cdot \frac{\partial^2}{\partial t^2} \left[ V_b(x,t) + V_s(x,t) \right]$$

$$= - \frac{\partial}{\partial x} \left[ G \cdot \Omega [b,h] \cdot \frac{\partial}{\partial x} V_s(x,t) \right] = Q(x,t)$$  \hspace{1cm} (7)
Furthermore, assuming that the material and other geometric parameters do not depend on \(x\) and \(t\), these two general differential equations can be combined into three new equations, describing the deflection due to bending \(V_b\) (equation 17), the deflection due to shear \(V_s\) (equation 18) and the total deflection \(V_t\) (equation 15).

For short writing we will adopt the subscripts \(x\) and \(t\) for the derivations with respect to the place \(x\) and the time \(t\).

First of all we will deduce the differential equation for the total deflection \(V_t=V_b+V_s\). Rewriting equations 7 and 8:

\[
\frac{\partial^2}{\partial x^2} V_t = V_{t_{xx}}
\]

Differentiations of these equations with respect to \(x\) and \(t\) give:

\[
\frac{\partial^2}{\partial x \partial t} V_t = V_{t_{xt}}
\]

Elimination of the derivations of \(V_s\) leads to:
Integration of equation 8 over x leads to (including an unknown function $H[t]$):

$$V_s = \frac{I}{G \cdot \Omega [b, h]} \left[ \rho \cdot I \cdot V_{b_{xx}} - E \cdot I \cdot V_{b_{xx}} + H [t] \right]$$  \hspace{1cm} \text{(16)}

Elimination of the deflection due to shear ($V_s$) in equation 7 leads to:

$$\rho \cdot b \cdot h \left[ \rho \cdot I \cdot V_{b_{xx}} - E \cdot I \cdot V_{b_{xx}} + H [t] \right] = Q (x, t)$$  \hspace{1cm} \text{(17)}

Elimination of the deflection due to bending ($V_b$) leads to:

$$\frac{+ E \cdot I}{\rho \cdot b \cdot h} \left[ Q_{xx} - \rho \cdot b \cdot h \cdot V_{s_{xx}} + G \cdot \Omega [b, h] \cdot V_{s_{xx}} \right]$$

$$\frac{- \rho \cdot I}{\rho \cdot b \cdot h} \left[ Q_{tt} - \rho \cdot b \cdot h \cdot V_{s_{tt}} + G \cdot \Omega [b, h] \cdot V_{s_{tt}} \right]$$

$$+ G \cdot \Omega [b, h] \cdot V_{s_{tt}} = H [t]$$  \hspace{1cm} \text{(18)}

These equations contain an unknown function $H[t]$. This implies that both $V_b$ and $V_s$ will contain a function, which only depends on the time $t$: $V_b :: f[t]$ and $V_s :: g[t]$.

Regarding equations (16), (17) and (18) it follows that:

$$G \cdot \Omega [b, h] \cdot f_{tt}, [t] + \rho \cdot I \cdot f_{tt}, [t] = - H_{tt}, [t]$$  \hspace{1cm} \text{(19)}

$$G \cdot \Omega [b, h] \cdot g_{tt}, [t] + \rho \cdot I \cdot g_{tt}, [t] = + H_{tt}, [t]$$  \hspace{1cm} \text{(20)}

$$G \cdot \Omega [b, h] \cdot g [t] = \rho \cdot I \cdot f_{tt}, [t] + H [t]$$  \hspace{1cm} \text{(21)}

Therefore, it yields that: $g[t] = - f[t]$. Because in the measurement itself, one will only measure the total deflection $V_t = V_b + V_s$, the function $H[t]$ can be taken equal to zero.

In view of the problem: "measuring the (dynamic) bending of a beam", the (unknown) function $H[t]$ can be taken equal to zero: $H[t] = 0.$

1.2 Four and three point bending

For the four point bending test it is assumed that the deflection due to shear (equation 18) can be
neglected and only the terms $\rho b h V_b t$ and $E I V_b \omega x$ (and of course $Q$) are important in differential equation 17. In this paragraph we will check the assumption for an asphalt beam, using common values for the dimensions of the beam and the asphalt material properties.

As will be shown later on, the applied force at the inner clamps (and the reaction forces at the outer clamps) can be transformed into a force distribution $Q(x,t)$ of the form:

$$Q(x,t) \propto F_0 \sin \left( n \cdot \frac{\pi}{L} \right) e^{i \omega_0 t}$$

Regarding differential equation 17 the deflection $V_b$ will be of the same form. In order to compare the importance of the several terms in the differential equation in a simple manner, we define the coefficient as $C_n/n^4$.

$$V_b(x,t) \propto \frac{C_n}{n^4} \sin \left( n \cdot \frac{\pi}{L} \right) e^{i \omega_0 t}$$

Invoking this (assumed) solution in differential equation 17 for $V_b$ leads to:

$$C_n \left[ \beta_1 \cdot \beta_2 \cdot \beta_3 + \beta_4 \cdot \beta_5 \right] \propto F_0$$

in which the coefficients $\beta_i$ are given by:

$$\begin{align*}
\beta_1 &= E \cdot I \cdot \frac{\pi^2}{L^2} ; \\
\beta_2 &= \rho \cdot b \cdot h \cdot \frac{\omega_0^2}{n^4} ; \\
\beta_3 &= \rho \cdot I \cdot \frac{\omega_0^2 \cdot \pi^2}{n^2 \cdot L^2} ; \\
\beta_4 &= \frac{\rho \cdot b \cdot h}{G \cdot \Omega [b,h]} \cdot \rho \cdot I \cdot \frac{\omega_0^2}{n^4} \\
\beta_5 &= \frac{\rho \cdot b \cdot h}{G \cdot \Omega [b,h]} \cdot E \cdot I \cdot \frac{\omega_0^2 \cdot \pi^2}{n^2 \cdot L^2} \\
\Omega [b,h] &\approx \frac{2}{3} b \cdot h ; \\
G &= \frac{E}{2 \cdot (1 + \nu)}
\end{align*}$$

To compare the 5 different $\beta$ values, the following values have been adopted:

<table>
<thead>
<tr>
<th>Table 1.1</th>
<th>Input figures for simulation calculations</th>
</tr>
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<tbody>
<tr>
<td>E [MPa]</td>
<td>L [m]</td>
</tr>
<tr>
<td>4000</td>
<td>0.4</td>
</tr>
</tbody>
</table>

In figure 1.1 the ratios of $\beta_i$ and $\beta_1$ are given as a function of the applied frequency $f_0 (= \omega_0/(2\pi))$ for $n=1$. In figure 1.2 these ratios are given for increasing $n$ values at the commonly used frequency of 30 Hz (in practice 29.3 Hz). As shown in the figures only the coefficients $\beta_1$ and $\beta_2$ are important. Finally in figure 1.3 the coefficient $C_n/n^4$ (based on $\beta_1$ and $\beta_2$) is given for increasing $n$ values at a frequency of 30 Hz.
Figure 1.1: Ratios $\beta_i/\beta_1$ as a function of the frequency $f_o$ for the first term $N=1$.

Figure 1.2: Ratios $\beta_i/\beta_1$ for increasing $N$ values at a frequency $f_o$ of 30 Hz.
Figure 1.3  The value of the coefficient $C_N/N^4$ for increasing $N$ values at $f_o=30$ Hz.

Using equation 16 the value of the shear deflection $V_s$ can be compared directly to $V_b$. In figure 1.4 the ratio of $V_s$ and $V_b$ is given for increasing $n$ values. As shown in the figure the shear deflection $V_s$ becomes greater with increasing $n$ values. However, the absolute values are negligible compared to the value of $V_b$ at $n=1$ (see figure 1.3).
Comparison of the coefficients in the differential equation for $V_s$ leads to the "approximated" differential equation:

$$G \cdot \Omega \ [b,h] \cdot V_{s,x} = -Q (x,t)$$

### 1.3 Conclusions

It can be concluded that for the (dynamic) bending of asphalt beams in a 4 point bending test\(^2\) the differential equations for the deflections $V_b$ (pure bending) and $V_s$ (shear) can be presented by:

$$E \cdot I \cdot V_{b,xx} + \rho \cdot b \cdot h \cdot V_{b,t} = Q (x, t)$$

$$G \cdot \Omega \ [b,h] \cdot V_{s,x} = -Q (x,t)$$

**It should be noted that although the influence of the deflection due to shear is small (4%) compared to the deflection due to pure bending, this has to be taken into account in case of calibration of an equipment.**

References


Thompson, “Vibration Theory and Applications”, 1974

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\(^2\) The same conclusion can also be drawn for the 3 point dynamic bending test
2. Static Bending

2.1 Solution using a series development in orthogonal sines

In the following text we will solve the static problem directly from the basic 4th order differential equation. This procedure is also used in the case of dynamic bending and forms the coupling between the dynamic and static problem. The latter one is commonly described by a second order differential equation. In this chapter we will only describe the situation in which the length $L$ between the outer clamps equals the total length $L_{tot}$ of the beam. In chapter 6 and 7 we discuss the problems if the beam is longer than this distance $L$.

The fourth order differential equation is given by:

$$E \cdot I \cdot \frac{d^4}{dx^4} V_b(x) = Q(x)$$  \hspace{1cm} (28)

A point load at a point $x_0$ can be developed into an infinite series of orthogonal functions on the interval $0 \leq x \leq L$. Bearing in mind the allowable deflection profiles for the beam ($V_b = 0$ at $x = 0$ and $x = L$) the orthogonal functions $\sin(n \cdot \pi \cdot x/L)$ are chosen. The development of the point load $F_A (= F_0/2)$ at the inner clamp A is then given by:

$$Q(x)_{\text{due to force } F_A} = \frac{2 \cdot F_A}{L} \cdot \sum_{n=1}^{\infty} \left[ \sin(n \cdot \pi \cdot A/L) \cdot \sin(n \cdot \pi \cdot x/L) \right]$$  \hspace{1cm} (29)

Mark that this is not a development into Fourier series. In that case the series will contain both $\sin(2 \cdot n \cdot \pi \cdot x/L)$ as well as $\cos(2 \cdot n \cdot \pi \cdot x/L)$ terms.

As the inner clamps A and B are situated symmetrical round $x=L/2$ it follows that with $F_B = F_A$:

$$\sin(n \cdot \pi \cdot B/L) = \sin(n \cdot \pi \cdot (L - A)/L) = -(-1)^n \cdot \sin(n \cdot \pi \cdot A/L)$$  \hspace{1cm} (30)

$$Q(x)_{\text{due to force } F_A + F_B} = \frac{2 \cdot F_A}{L} \cdot \sum_{n=1}^{\infty} \left[ \frac{1 - (-1)^n}{2} \cdot \sin(n \cdot \pi \cdot A/L) \cdot \sin(n \cdot \pi \cdot x/L) \right]$$  \hspace{1cm} (31)

$$= \frac{2 \cdot F_A}{L} \cdot \sum_{n=1,3,5,...}^{\infty} \left[ \frac{1 - \cos(n \cdot \pi)}{2} \cdot \sin(n \cdot \pi \cdot A/L) \cdot \sin(n \cdot \pi \cdot x/L) \right] = Q(x) \{\text{total}\}$$  \hspace{1cm} (32)

Notice that the reaction forces at $x = 0$ and $x = L$ are transformed into a zero force distribution $Q(x) : \sin(n \cdot \pi \cdot 0/L) = \sin(n \cdot \pi \cdot L/L) = 0$. This is not the case if the length of the beam ($L_{tot}$) is longer than the distance $L$ between the two outer clamps (see chapter 6).
Instead of the coordinate $A$ it is also possible to express the equation in the distance $a = B - A$ between the two inner clamps, which leads to:

$$
\frac{2 \cdot F_A}{L} \sum_{n=1}^{\infty} \left[ \sin \left( \frac{n \cdot \pi}{2} \right) \cdot \cos \left( n \cdot \pi \cdot \frac{a}{2 \cdot L} \right) \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L} \right) \right]
$$

The general solution for the deflection $V_b(x)$ can now be presented as the sum of a particular solution of the differential equation and the general solution of the homogenous differential equation ($Q(x)=0$). This last one is an ordinary third order polynomial. The coefficients are determined by the requirements for the boundary conditions:

$$
V_b(\ x \ ) = C_3 \cdot x^3 + C_2 \cdot x^2 + C_1 \cdot x + C_0 + \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot E \cdot I} \times \sum_{n=1}^{\infty} \left[ \frac{1 - \cos \left( n \cdot \pi \right)}{2 \cdot n^4} \cdot \sin \left( n \cdot \pi \cdot \frac{A}{L} \right) \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L} \right) \right]
$$

Because at $x=0$ and $x=L$ the second derivation of $V_b$ (; Moment) must be zero it follows that $C_3$ and $C_2$ are zero.

The deflection $V_s(x)$ can be expressed as:

$$
V_s(\ x \ ) = D_1 \cdot x + D_0 + \frac{2 \cdot F_0 \cdot L}{\pi^2 \cdot G \cdot \Omega \ [b, h]} \times \sum_{n=1}^{\infty} \left[ \frac{1 - \cos \left( n \cdot \pi \right)}{2 \cdot n^2} \cdot \sin \left( n \cdot \pi \cdot \frac{A}{L} \right) \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L} \right) \right]
$$

which is in accordance with $V_b$ if $D_1$ is zero (see equation 16)? Because at the coordinates $x=0$ and $x=L$ the total deflection must be zero it follows that $C_1$ is zero and $C_0 = -D_0$. This means that the total deflection $V_t=V_b+V_s$ does not contain a constant.

Therefore in the case of static bending the general solution of the homogenous differential equation can be ignored.
2.2 Summation of the infinite series

The infinite series for \( V_b(x) \) and \( V_s(x) \) are sum able by expressing the product of the nominator as a finite sum of \( \cos(n \cdot \varphi) \) terms. The following procedure has to be performed:

\[
\frac{1}{2} \cdot \sin \left( n \cdot \pi \cdot \frac{A}{L} \right) \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L} \right) = \cos \left( n \cdot \frac{x - A}{L} \right) \cdot \cos \left( n \cdot \frac{x + A}{L} \right) = \cos \left( n \cdot \varphi_1 \right) - \cos \left( n \cdot \varphi_2 \right)
\]

\[
\frac{4}{4} \cdot \cos \left( n \cdot \pi \right) \cdot \cos \left( n \cdot \varphi_1 \right) = \frac{1}{4} \cdot \cos \left( n \cdot \frac{x - A}{L} \right) \cdot \cos \left( n \cdot \frac{x + A}{L} \right) = \cos \left( n \cdot \varphi_3 \right) + \cos \left( n \cdot \varphi_4 \right)
\]

\[
\frac{8}{8} \cdot \cos \left( n \cdot \pi \cdot \varphi_1 \right) \cdot \cos \left( n \cdot \varphi_4 \right) = \frac{1}{4} \cdot \cos \left( n \cdot \pi \right) \cdot \cos \left( n \cdot \varphi_2 \right) = \cos \left( n \cdot \varphi_5 \right) + \cos \left( n \cdot \varphi_6 \right)
\]

Furthermore it is proven that these types of series are sum able:

\[
\sum_{n=1}^{\infty} \left[ \frac{\cos \left( n \cdot \varphi \right)}{n^4} \right] = \frac{\pi^4}{90} - \frac{\pi^2 \cdot \varphi^2}{12} + \frac{\pi \cdot \varphi^3}{12} - \frac{\varphi^4}{48}
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{\cos \left( n \cdot \varphi \right)}{n} \right] = - \log \left( 2 \cdot \sin \left( \frac{\varphi}{2} \right) \right)
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{\cos \left( n \cdot \varphi \right)}{n^2} \right] = \frac{\pi^2}{6} - \frac{\pi \cdot \varphi}{2} + \frac{\varphi^2}{4}
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{\sin \left( n \cdot \varphi \right)}{n} \right] = \frac{1}{2} \cdot [\pi \cdot \varphi]
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{\sin \left( n \cdot \varphi \right)}{n^3} \right] = \frac{\pi^2 \cdot \varphi}{6} - \frac{\pi \cdot \varphi^2}{4} + \frac{\varphi^3}{12}
\]

\[
\sum_{n=1}^{\infty} \left[ \frac{\sin \left( n \cdot \varphi \right)}{n^5} \right] = \frac{\pi^4 \cdot \varphi}{90} - \frac{\pi^2 \cdot \varphi^3}{36} + \frac{\pi \cdot \varphi^4}{48} - \frac{\varphi^5}{240}
\]
In this way one will obtain for each \( x \) value an analytical algebraic expression for the deflection. For special \( x \) values another easier solution procedure can be obtained by:

\[
\sum_{n=1, 3, 5}^{\infty} \frac{F[n]}{n^p} = \sum_{n=1}^{\infty} \frac{F[n]}{n^p} - \frac{1}{2^p} \sum_{n=1}^{\infty} \frac{F[2n]}{n^p}
\]

which in case of \( p = 4 \) leads to:

\[
V_b(x) = \sum_{n=1}^{\infty} \frac{\sin(n \cdot \pi \cdot \frac{A}{L}) \cdot \sin(n \cdot \pi \cdot \frac{x}{L})}{n^4}
\]

\[
\frac{1}{2^4} \sum_{n=1}^{\infty} \frac{\sin(2 \cdot n \cdot \pi \cdot \frac{A}{L}) \cdot \sin(2 \cdot n \cdot \pi \cdot \frac{x}{L})}{n^4}
\]

For \( x = L/2 \) the second series is zero for each term, while for the combination of \( x = L/3 \) and \( A = L/3 \) it yields that the second sum equals \( 1/16 \times \) the first sum:

\[
\sin^2(n \cdot \pi \cdot \frac{2}{3}) = \frac{1 - \cos(n \cdot \pi \cdot \frac{4}{3})}{2} = \frac{1 - \cos(2 \cdot n \cdot \pi - n \cdot \pi \cdot \frac{2}{3})}{2} = \sin^2(n \cdot \pi \cdot \frac{1}{3})
\]

The same procedure can also be carried out for the equations in which instead of the coordinate \( A \) the distance \( \alpha \) between the two inner clamps is used.

---

**A remark should be made for the 'validity' range of the sum able series.**

The given expressions for the sum able series are only valid if the value \( \varphi \) is in the interval \( 0 \leq \varphi \leq 2 \times \pi \).

So, calculating the moments along the beam, using the series expressions for the deflections, it is important to consider the cases \( x \leq A \), and \( A \leq x \leq (L - A) \) separately.

---

Following the procedures given above, it will be shown later on that the dynamic solution, which is also expressed in a sum of orthogonal sine’s, reduces to the static solution if the frequency \( \omega_0 \) goes to zero.
2.3 The common second order differential equation in static bending

In this paragraph we will give the derivation of the second order differential equation which is more familiar in case of static bending.

For the bending of a rectangular beam the basic differential equations are given by:

\[-G x \Omega [b,h] \cdot \frac{d^2}{dx^2} V_s(x) = Q(x)\]  \[48\]

\[E \cdot I \cdot \frac{d^3}{dx^3} V_b(x) = -G x \Omega [b,h] \cdot \frac{d}{dx} V_s(x)\]  \[49\]

in which \(V_s\) is the deflection due to shear and \(V_b\) is the deflection due to bending.

Elimination of \(V_s\) leads to the well known fourth order differential equation:

\[E \cdot I \cdot \frac{d^4}{dx^4} V_b(x) = Q(x)\]  \[50\]

The load \(Q(x)\) consists out of two applied point loads \(F_A\) and \(F_B\) (=\(F_A=\frac{F_0}{2}\)) at the clamps at \(x=A\) and \(x=B\) and two reaction loads - \(F_B/2\) at the clamps at \(x=0\) and \(x=L\).

\[Q(x) = F_0/2 \times \left[ \delta(x = A) + \delta(x = B) - \delta(x = 0) - \delta(x = L) \right]\]  \[51\]

To avoid pseudo-mathematical problems, we will start with a beam of length \(L+2\Delta\) and when the solution is obtained we will take \(\Delta\) equal to 0. Therefore the boundary conditions are:

\[V_b(0) = V_b(L) = \frac{d^2}{dx^2} V_b(x)_{x=\pm \Delta} = \frac{d^2}{dx^2} V_b(x)_{x=L\pm \Delta} = 0\]  \[52\]

Integration of equation 50 for the interval from \(-\Delta\) to \(L+\Delta\) gives:

\[E \cdot I \left[ \frac{d^3}{dx^3} V_b(x)_{x=L+\Delta} - \frac{d^3}{dx^3} V_b(x)_{x=-\Delta} \right] = 0\]  \[53\]

Because it is a symmetrical problem around the centre \(x=L/2\) it yields that:

\[\frac{d^3}{dx^3} V_b(x)_{x=\pm \Delta} = - \frac{d^3}{dx^3} V_b(x)_{x=L\pm \Delta}\]  \[54\]

\[\rightarrow E \cdot I \cdot \frac{d^3}{dx^3} V_b(x)_{x=\pm \Delta} = 0\]

Integration of equation 50 from \(x=-\Delta\) to \(x=x\) (upper limit: \(-\Delta < x < 0\)) gives:

\[E \cdot I \left[ \frac{d^3}{dx^3} V_b(x) - \frac{d^3}{dx^3} V_b(x)_{x=-\Delta} \right] = 0 \rightarrow E \cdot I \cdot \frac{d^3}{dx^3} V_b(x) = 0\]  \[55\]

Again integration of equation 55 from \(x=-\Delta\) to \(x=x\) (upper limit: \(-\Delta < x < 0\)): 
\[ E \cdot I \cdot \left[ \frac{d^2}{dx^2} V_b(x) - \frac{d^2}{dx^2} V_b(x) \right]_{x=\Delta} = E \cdot I \cdot \frac{d^2}{dx^2} V_b(x) = 0 \]  

Bearing in mind that at \( x=0 \) the deflection \( V_b=0 \), the solution of this equation is:

\[ V_b(x) = C_1 \cdot x \]

The second interval for the upper limit (0<\( x <A \)) includes the reaction force at \( x=0 \). Integration from \( x=-\Delta \) to \( x=x \) with for the upper limit: \( 0 < x < A \) leads to:

\[ E \cdot I \cdot \left[ \frac{d^3}{dx^3} V_b(x) - \frac{d^3}{dx^3} V_b(x) \right]_{x=\Delta} = - \frac{F_0}{2} \rightarrow \frac{d^3}{dx^3} V_b(x) = - \frac{F_0}{2 \cdot E \cdot I} \cdot x \]

Integration of equation 58 from \( x=0 \) to \( x=x \) with the upper limit between \( 0 < x < A \) gives:

\[ \frac{d^2}{dx^2} V_b(x) - \frac{d^2}{dx^2} V_b(x) \left|_{x=0} \right. = \frac{d^2}{dx^2} V_b(x) = - \frac{F_0}{2 \cdot E \cdot I} \cdot x \]

Notice the lower limit \( x=0 \) of the integral. From equation 57 it follows that the second derivation is zero at \( x=0 \) (the moments are continuous along the beam).

Now we consider the third interval \( A < x < B (=L-A) \). We will integrate the 4th order differential equation from \( x=-\Delta \) to \( x=x \) with \( A < x < B \). Therefore both the reaction force at \( x=0 \) and the applied force at \( x=A \) are included. The integration leads to:

\[ E \cdot I \cdot \left[ \frac{d^3}{dx^3} V_b(x) - \frac{d^3}{dx^3} V_b(x) \right]_{x=A} = - \frac{F_0}{2} + \frac{F_0}{2} = 0 \]

\[ \rightarrow E \cdot I \cdot \frac{d^3}{dx^3} V_b(x) = 0 \]

The obtained third order differential equation is only valid on the third interval: \( A < x < B \). Integration of this equation from \( x=A \) to \( x=x \) with \( A < x < B \) gives:

\[ E \cdot I \cdot \left[ \frac{d^2}{dx^2} V_b(x) - \frac{d^2}{dx^2} V_b(x) \right]_{x=A} = 0 \rightarrow E \cdot I \cdot \frac{d^2}{dx^2} V_b(x) = - M_0 \]

Because \( d^2V/dx^2 \) has to be continuous at \( x=A \) it follows that:

\[ - E \cdot I \cdot \frac{d^2}{dx^2} V_b(x) = \frac{F_0}{2} \cdot A = M_0 \]

So, for the whole interval -\( \Delta < x < L+\Delta \) three second order differential equations are obtained. Mark that for the interval 0<\( x <L \) the two obtained differential equations do not depend on the distance \( \Delta \). Therefore in static bending the real length \( L_{tot} \) does not influence the bending of the beam on the interval \( L \) between the two outer clamps. So, the third differential equation for the
interval \(-\Delta < x < 0\) is often ignored in text books.
2.4 Solution of the second order differential equation

2.4.1 General

First of all we will describe the solution as given in many text books using a different coordinate system. The deflection $V_b(x)$ along the beam is determined for pure bending by the following second order differential equation and boundary conditions:

$$-EI \cdot \frac{d^2 V_b(x)}{dx^2} = M(x)$$

**Region I**: $0 \leq x \leq \frac{a}{2}$

$$M(x) = M_0 = \frac{F_0}{2} \cdot \left[ \frac{L}{2} - \frac{a}{2} \right]$$

**Region II**: $\frac{a}{2} \leq x \leq \frac{L}{2}$

$$M(x) = \frac{F_0}{2} \cdot \left[ \frac{L}{2} - x \right]$$

$$x = 0: \quad \frac{dV_b(x)}{dx} = 0 \quad ; \quad x = \frac{L}{2}: \quad V_b(x) = 0$$

$$x = \frac{a}{2}: \quad \frac{dV_b(x)}{dx} \quad \text{and} \quad V_b(x) \quad \text{are continuous}$$

**Region I**

$$V_b(x) = -\frac{F_0}{96EI} \cdot \left[ 12x^2 - (2.2L^2 + 2.2a.2L - a^2) \right]$$

**Region II**

$$V_b(x) = -\frac{F_0}{96EI} \cdot \left[ -8x^3 + 12.2Lx^2 - 6.2a^2x + 3.2.2L - 2.2L \right]$$

Instead of the coordinate system above, the formulas can also be expressed using our coordinate system $0 < x < L$. In case of dynamic bending the formulas are easier to express in that coordinate system. Also the length $a$ between the inner two clamps is replaced in that case by the coordinate $A$ of the first inner clamp (see figure 2.0).
Figure 2.0: Our coordinate system $0 < x < L$ and the system $0 < x' < L/2$

The relation between $a$ and $A$ is: $a = L - 2A$. With these substitutions the formulas can be rewritten in our coordinate system as:

$$
\begin{align*}
\text{Region I: } & A \leq x \leq \frac{L}{2}: \quad M(x) = \frac{F_0}{2} \cdot A \\
\text{Region II: } & 0 \leq x \leq A: \quad M(x) = \frac{F_0}{2} \cdot x
\end{align*}
$$

$$
\begin{align*}
\text{Region I: } & V(x) = \frac{8F_0L^3}{96EI} \cdot A \left[ + \frac{x}{L} - 3 \frac{x^2}{L^2} - \frac{A^2}{L^2} \right] \\
\text{Region II: } & V(x) = \frac{8F_0L^3}{96EI} \cdot x \left[ + \frac{A}{L} - 3 \frac{A^2}{L^2} - \frac{x^2}{L^2} \right]
\end{align*}
$$

Mark the resemblance ($x \leftrightarrow A$) between the two regions.

Both coordinate systems can be used in the formulas.
Preference is given to the coordinate system $0 < x < L$. If it is not explicitly mentioned the formulas will be based on this coordinate system.

In case $\Delta > 0$ (chapter 6 and 7) the coordinate system starts at the beginning of the beam. The $x$ coordinate of the first clamp will be $x = \Delta$.

Therefore $V_{\text{max}}$ in the center of the beam may be presented as:

$$
V_{\text{max}} = \frac{F_0 \cdot (L - a)}{96EI} \left[ 2 \cdot L^2 + 2 \cdot a \cdot L - a^2 \right]
$$

or as:

$$
V_{\text{max}} = \frac{2 \cdot F_0 \cdot A}{96EI} \left[ 3 \cdot L^2 - 4 \cdot A^2 \right]
$$
2.4.2 **Maximum strain and stress in the beam**

The following (general) relation exists between the (horizontal) strain in the beam and the deflection:

\[
\varepsilon(\mathbf{x}, y) = -\frac{M(\mathbf{x}) \cdot y}{E \cdot I} \rightarrow \varepsilon_{\text{max}} = -\frac{M_{\text{max}} \cdot h}{2 \cdot E \cdot I} \rightarrow \frac{M_{\text{max}}}{E \cdot x \cdot I} = \frac{2 \cdot \varepsilon_{\text{max}}}{h}
\]

For region I the strain is given by:

\[
I : \varepsilon(x, \frac{h}{2}) = \frac{F_0 \cdot A}{E \cdot I} \cdot \frac{h}{4}
\]

and for region II:

\[
II : \varepsilon(x, \frac{h}{2}) = \frac{F_0 \cdot x \cdot h}{E \cdot I} \cdot \frac{h}{4}
\]

Using the maximum deflection \( V_{\text{max}} \) in the center of the beam, the maximal strain \( \varepsilon_{\text{max}} \) is given by:

\[
V_{\text{max}} = \varepsilon_{\text{max}} \cdot \frac{2 \cdot L^2 + 2 \cdot a \cdot L - a^2}{12 \cdot h} = \varepsilon_{\text{max}} \cdot \frac{3 \cdot L^2 - 4 \cdot A^2}{12 \cdot h}
\]

The maximum bending stress is given by:

\[
\sigma_{\text{max}} = E \cdot \varepsilon_{\text{max}} = \frac{F_0 \cdot (L - a) \cdot h}{2 \cdot b \cdot h^3} = \frac{3 \cdot F_0 \cdot (L - a)}{2 \cdot b \cdot h^2} \equiv \frac{3 \cdot F_0 \cdot A}{b \cdot h^2}
\]

The expression for the stiffness modulus of the beam is given by:

\[
E = \frac{\sigma_{\text{max}}}{\varepsilon_{\text{max}}} = \frac{(L - a) \cdot (2 \cdot L^2 + 2 \cdot a \cdot L - a^2)}{8 \cdot b \cdot h^3} \cdot \frac{F_0}{V_{\text{max}}} = \left\langle \frac{2 \cdot A \cdot [3 \cdot L^2 - 4 \cdot A^2]}{8 \cdot b \cdot h^3}, \frac{F_0}{V_{\text{max}}} = Z \cdot \frac{F_0}{V_{\text{max}}} \right\rangle
\]

2.4.3 **Mean Deflection**

The mean deflection:

\[
V(\mathbf{x}) = \frac{1}{L/2} \int_{x=0}^{L/2} V(\mathbf{x}) \, d\mathbf{x} = \frac{5 \cdot L^4 - 6 \cdot L^2 \cdot a^2 + a^4}{4 \cdot L} \cdot \frac{F_0}{9 \cdot 6 \cdot E \cdot I}
\]

**REMARK:**

It should be noticed that in the case of static bending the mass of the beam is not relevant. This implies that the actual length of the beam is not important but only the length between the outer clamps. In dynamic bending tests the pieces of the beam outside the outer clamps are relatively important, because 1) the mass plays a role in the fourth order differential equation and 2) in that case the moments at the ends of the actual beam are only zero and not also at the outer clamps.
2.4.4 Interpretation procedure for determining the Young's modulus \( E \), maximum strain and stress from static bending tests.

The common procedure in the (static) measurements will be the following steps:

1) Measurement of the force \( F_0 \) and the maximum displacement \( V_{\text{max}} \).

   Calculation of the displacement \( V(x) \) along the beam:

   \[
   \text{Region I: } V(x) = \frac{4 \cdot [3 \cdot x \cdot L - 3 \cdot x^2 - A^2]}{3 \cdot L^2 - 4 \cdot A^2} \cdot V_{\text{max}}
   \]

   \[
   \text{Region II: } V(x) = \frac{4 \cdot [3 \cdot A \cdot L - 3 \cdot A^2 - x^2]}{3 \cdot L^2 - 4 \cdot A^2} \cdot \frac{x}{A} \cdot V_{\text{max}}
   \]

2) Calculation of the \( E \) value by:

   \[
   E = Z \cdot \frac{F_0}{V_{\text{max}}} \quad \text{with} \quad Z = \frac{2 \cdot A \cdot [3 \cdot L^2 - 4 \cdot A^2]}{8 \cdot b \cdot h^3}
   \]

3) Calculation of the strain at \( y=h/2 \) along the beam:

   \[
   \text{Region I: } \varepsilon(\frac{x}{2}, \frac{h}{2}) = \frac{3 \cdot h}{3 \cdot x \cdot L - 3 \cdot x^2 - A^2} \cdot V(x)
   \]

   \[
   \text{Region II: } \varepsilon(\frac{x}{2}, \frac{h}{2}) = \frac{3 \cdot h}{3 \cdot A \cdot L - 3 \cdot A^2 - x^2} \cdot V(x)
   \]

Calculation of the maximum strain at \( x=L/2 \) and \( y=h/2 \):

\[
\varepsilon_{\text{max}} = \frac{12 \cdot h}{3 \cdot L^2 - 4 \cdot A^2} \cdot V_{\text{max}}
\]

4) Calculation of the maximum stress according to:

\[
\sigma_{\text{max}} = E \cdot \varepsilon_{\text{max}} = \frac{12 \cdot Z \cdot h}{3 \cdot L^2 - 4 \cdot A^2} \cdot F_0 = \frac{3 \cdot A}{b \cdot h^2} \cdot F_0
\]

The same type of procedure is followed in the case of dynamic bending (of a viscous-elastic material).
2.4.5  **ASTM Specifications**

In a bending test according to the ASTM specifications, the distance \( a \) between the two inner clamps is \( L/3 \). So, the maximum value for the deflection in the center is:

\[
V_{\text{max}} = V\left(\frac{L}{2}\right) = \frac{138}{81} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.7037037 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

The deflection at the inner clamp is:

\[
V(A) = \frac{120}{81} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.4814815 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

The mean deflection is given by:

\[
\bar{V}(x) = \frac{88}{81} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.0864198 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

2.4.6  **DWW Specifications**

In the DWW specifications the length between the outer clamps is 400 mm and the length between the two inner clamps is 130 mm. The total length of the actual beam is about 450 mm. With these figures one will obtain the following expressions:

\[
V_{\text{max}} = V\left(\frac{L}{2}\right) = \frac{27 \cdot 4071}{40 \cdot 1600} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.7174531 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

\[
V(A) = \frac{27 \cdot 891}{40 \cdot 400} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.5035625 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

\[
\bar{V}(x) = \frac{11206161}{10240000} \cdot \frac{F \cdot L^3}{96 \cdot E \cdot I} = 1.0943517 \times \frac{F \cdot L^3}{96 \cdot E \cdot I}
\]

The differences between the DWW equipment and the ASTM requirements are small in case of static bending.
2.5 First order approximation of the solution expressed in an infinite series of sines

Because of the small contributions of the higher order terms in this type of solutions, it seems allowed to take (in case of the deflection) only the first term of the series (equation 34). This substitution is based on the following approximations:

\[
Ratio \ I = \frac{8}{96} \cdot \frac{A}{L} \cdot \left[ 3 \cdot \frac{x}{L} - 3 \cdot \frac{x^2}{L^2} - \frac{A^2}{L^2} \right] \approx 1; \ A < x < \frac{L}{2}
\]

\[
Ratio \ II = \frac{8}{96} \cdot \frac{x}{L} \cdot \left[ 3 \cdot \frac{A}{L} - 3 \cdot \frac{A^2}{L^2} - \frac{x^2}{L^2} \right] \approx 1; \ 0 < x < A
\]

In the following figures the ratios of these approximations are given as a function of x/L for two A/L values (figures 2.1 and 2.2) and as a function of A/L for two values of x/L (figures 2.3 and 2.4). As shown in the first figure, if A/L is about 1/3 (ASTM specifications) the approximation is quite good (the error is less than 0.3 % on the interval A/L < x/L < 1/2). This is due to the fact that for A/L=1/3 the second term (n=3) vanishes in the series development. For A/L=1/10 the approximation is worse.

Figure 2.1 The value of Ratios I and II as a function of x/L for A/L=1/3
Figure 2.2  The value of Ratios I and II as a function of x/L for A/L=1/10

The figures 2.3 and 2.4 give the ratios at the center (x=L/2) and at the inner clamp (x=A/L) as a function of A/L. Although the approximation itself becomes worse for smaller A/L values, the obtained deflection profiles are still acceptable as shown in the figures 2.5 and 2.6.
Figure 2.3  The Ratios I and II as a function of $A/L$ for the center $x=L/2$

Figure 2.4  The Ratios I and II as a function of $A/L$ for the inner clamp $x=A$

Figure 2.5  The correct deflection profiles (Static 1 and 2) compared to the approximated profile by the first term only ("Dynamic") for $A/L=1/3$. 
Figure 2.6  The correct deflection profile (Static 1 and 2) compared to the approximated profile by the first term ("Dynamic") for A/L=1/10.

So, in static bending test in which A is about L/3, it is allowed for practical purposes to take either the (correct) static solution or only the first term of the solution which is expressed as a (infinite) sum of orthogonal sine’s for the deflection profile.

The "substitution" is less good for the calculations of the moments (and strains). Here the following approximation is used:

\[ Ratio \text{ III} = \frac{A}{2} \cdot \frac{L}{\pi^2} \cdot \sin \left( \frac{\pi}{L} \cdot \frac{x}{A} \right) \cdot \sin \left( \frac{\pi}{L} \cdot \frac{A}{L} \right) \approx 1 ; \text{ for } A < x < \frac{L}{2} \]

\[ Ratio \text{ IV} = \frac{x}{2} \cdot \frac{L}{\pi^2} \cdot \sin \left( \frac{\pi}{L} \cdot \frac{x}{A} \right) \cdot \sin \left( \frac{\pi}{L} \cdot \frac{A}{L} \right) \approx 1 ; \text{ for } 0 < x < A \]

In the figures 2.7 and 2.8 the ratios of these approximations are given as a function of x/L for two A/L values. As shown in figure 2.7, if A/L is about 1/3 this approximation is still quite acceptable.
on the interval $A/L < x/L < 1/2$. However, if $A/L=1/10$ the approximation becomes very bad.

**Figure 2.7** Ratios III and IV as functions of $x/L$ for $A/L=1/3$

**Approximation**

$A/L = 1/3$

![Graph showing Ratio III and IV as functions of $x/L$ for $A/L=1/3$.](image)

**Approximation**

$A/L = 1/10$

![Graph showing Ratio III and IV as functions of $x/L$ for $A/L=1/10$.](image)
Figure 2.8  Ratios III and IV as functions of x/L for A/L=1/10

The figures 2.9 and 2.10 give an impression of the normalized moment profile according to the correct static solution (static 1 and static 2) and the calculated profile ("dynamic") based on the first term of the static solution expressed in an infinite series of orthogonal sine’s.

![Approximation Diagram](image)

Figure 2.9  Comparison of the (normalized) moment profile according to the first order approximation ("Dynamic") and the correct profile (Static 1 & 2) for A/L=1/3.
Figure 2.10  
Comparison of the (normalized) moment profile according to the first order approximation ("Dynamic") and the correct profile (Static 1 & 2) for A/L=1/10.

The following conclusions can be made:

1)  In case of determining the E values by static tests it is allowed to use the first order approximation of the static solution expressed in an infinite series of orthogonal sines.

2)  It is not advised to use this approximation for the calculation of the moments, strains and stresses in the beam.

In chapter 8 we will go further into detail for the consequences of the first order approximation.
3. Dynamic Bending of an Elastic Beam  
(Accounting only for the mass of the beam)

3.1 General

The differential equation for pure dynamic bending is given by:

\[ E \cdot I \cdot \frac{\partial^4 V(x, t)}{\partial x^4} + \rho^* \cdot \frac{\partial^2 V(x, t)}{\partial t^2} = Q(x, t) \]  

with \( \rho^* = \rho \cdot B \cdot h = M_{\text{beam}}/L_{\text{tot}} \cdot \text{kg/m} \); \( L_{\text{tot}} = \text{actual length of the beam} \) (\( L_{\text{tot}} = L+2\Delta \)).

The coordinate system is chosen from the beginning of the beam (\( x=0 \)). The coordinates of the clamps are: \(+\Delta, +\Delta + A, +\Delta + B\) and \(+L_{\text{tot}} - \Delta\).

The (total) applied force \( F(x, t) = F_0 \cdot \sin(\omega_0 \cdot t) \) is divided over the two inner clamps (\( x=\Delta+A \) and \( x=\Delta+B \)): \( F_A \cdot \sin(\omega_0 \cdot t + \varphi_A) + F_B \cdot \sin(\omega_0 \cdot t + \varphi_B) = F_0 \cdot \sin(\omega_0 \cdot t) \).

The sum of the reaction forces at the two outer clamps at \( x = +\Delta \) and at \( x = L_{\text{tot}} - \Delta \) is equal to \(-F_0 \cdot \sin(\omega_0 \cdot t)\). The force distribution \( Q[x, t] \) along the beam is given by:

\[
Q(x, t) = F \cdot \delta(x = +\Delta) + F_A(t) \cdot \delta(x = +\Delta + A) + F_B(t) \cdot \delta(x = +\Delta + B) + F_{\text{tot} - \Delta}(t) \cdot \delta(x = +L_{\text{tot}} - \Delta)
\]

The boundary conditions are (notice the different origin of the X coordinate):

\[
\begin{align*}
\text{at}\ x = +\Delta &; \ t = t : V (+\Delta, t) = 0 \\
\text{at}\ x = 0 &; \ t = t : \frac{\partial^2 V(x, t)}{\partial x^2} |_{x=0} = 0 \\
\text{at}\ x = L_{\text{tot}} - \Delta &; \ t = t : V (L_{\text{tot}} - \Delta, t) = 0 \\
\text{at}\ x = L_{\text{tot}} &; \ t = t : \frac{\partial^2 V(x, t)}{\partial x^2} |_{x=L_{\text{tot}}} = 0
\end{align*}
\]

This differential equation is quite nasty; it will be discussed in chapters 6 and 7. If the actual length \( L_{\text{tot}} \) equals the length \( L \) between the two outer clamps (\( \Delta=0 \)), an easy analytical solution can be obtained. In that case the Eigen frequencies of the beam (free vibrations; resonance frequencies) are given by:

\[ \omega_n = \frac{n^2 \cdot \pi^2}{L^2} \cdot \sqrt{\frac{E \cdot I}{\rho^*}} = \frac{n^2 \cdot \pi^2}{L^2} \cdot \sqrt{\frac{E \cdot I}{M_{\text{beam}} \cdot L^3}} \]

The solution of the homogenous differential equation (\( Q=0 \)) is then given by:

\[ V(x, t) = \sum_{n=1}^{\infty} D_n \cdot \sin(n \cdot \pi \cdot \frac{x}{L}) \cdot \sin(\omega_n \cdot t - \varphi_n) \]
Of course the additional requirement for the deflections at the inner clamps reduces the allowable n values for the final solution: \( V(A,t) = V(B,t) \).

These solutions are called the free vibrations of the beam. In the absence of an external force (with frequency \( \omega_0 \)) these vibrations will occur if e.g. at \( t=0 \) the (elastic) beam has a given deflection profile.

Next to these solutions of the homogenous differential equation, we are also interested in solutions for which the frequency is prescribed: \( \omega = \omega_0 \). So, solutions for \( V(x,t) \) of the form:

\[
V(x,t) = V(x) \sin(\omega_0 t)
\]

In order to be a valid solution the function \( V(x) \) has to meet the following differential equation:

\[
E \cdot I \frac{d^4}{dx^4} V(x) - \rho \cdot \omega_0^2 \cdot V(x) = 0
\]

The general solution of this differential equation is given by:

\[
V(x) = C_1 \cdot e^{\beta x} + C_2 \cdot e^{\beta x} + C_3 \cdot e^{\beta x} + C_4 \cdot e^{-\beta x}
\]

with \( \beta = \sqrt[4]{\frac{\rho \cdot \omega_0^2}{E \cdot I}} \)

This can be used in order to meet the boundary conditions, if necessarily. As shown later on this is not the case in this chapter (elastic beam; \( L_{tot} = L \)).

This solution is only needed in case the actual length \( L_{tot} \) is longer than the distance \( L \) between the two outer clamps (\( \Delta > 0 \)). This problem will be discussed in chapter 6 for the elastic case. In this chapter we only deal with the problem in which \( L_{tot} = L \) (\( \Delta = 0 \)).
3.2 Dynamic solution in case $L_{tot} = L$

The two applied forces $F_A$ and $F_B$ at the two inner clamps can be developed into orthogonal series on the interval:

$$
F(x, t) = \frac{2 \cdot F_A \sin(\omega_0 t + \varphi_A)}{\rho L} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \right] + \frac{2 \cdot F_B \sin(\omega_0 t + \varphi_B)}{\rho L} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \right]$$

(102)

The reaction forces at the two outer clamps do not play a role because the transformation of these forces into orthogonal sine series leads to zero: $\sin(n \frac{\pi 0}{L}) = \sin(n \frac{\pi L}{L}) = 0$. This is not the case in the problem in which the actual length $L_{tot}$ is longer than the distance $L$ (see chapter 6).

Invoking these expressions in the fourth order differential equation, the deflection $V(x,t)$ can also be expressed as functions of these orthogonal series. Notice that at $x=0$ and $x=L$ these orthogonal series fulfill the boundary conditions. Therefore we do not need general solutions of the homogenous differential equation in order to meet the requirements for the boundary conditions.

$$
V(x, t) = \frac{2 \cdot F_A \sin(\omega_0 t + \varphi_A)}{\rho L} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \right] + \frac{2 \cdot F_B \sin(\omega_0 t + \varphi_B)}{\rho L} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \right] = \frac{2 \cdot F_A \cdot L^3 \sin(\omega_0 t + \varphi_A)}{\pi^4 \cdot E \cdot I} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{A}{L})} \right] n^4 \left[ 1 - \frac{\omega_0^2}{\omega_n^2} \right] + \frac{2 \cdot F_B \cdot L^3 \sin(\omega_0 t + \varphi_B)}{\pi^4 \cdot E \cdot I} \sum_{n=1}^{\infty} \left[ \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \cdot \frac{\sin(n \cdot \pi \frac{x}{L})}{\sin(n \cdot \pi \frac{B}{L})} \right] n^4 \left[ 1 - \frac{\omega_0^2}{\omega_n^2} \right]$$

(103)

Because the inner clamps are placed symmetrical round the center ($x=L/2$) it follows that:

$$
\sin(n \cdot \pi \frac{B}{L}) = \sin(n \cdot \pi \frac{L-A}{L}) = -( -1)^n \sin(n \cdot \pi \frac{A}{L})
$$

(107)
The additional requirement is that the deflections at x=A and x=B are equal and that 
\( F_A \cdot \sin(\omega \cdot t + \varphi_A) + F_B \cdot \sin(\omega \cdot t + \varphi_B) = F_0 \cdot \sin(\omega \cdot t) \). Thus only odd numbers are possible 
(n=1, 3, 5, 7, etc.), and numbers of n for which \( n \cdot \pi \cdot A/L \) has an integer value (\( \sin(n \cdot \pi \cdot A/L) = 0 \)).

Therefore the deflection can be given by:

\[
V(x, t) = \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot E \cdot I} \times 
\sum_{n=1}^{\infty} \frac{1 - \cos(n \cdot \pi)}{2} \cdot \sin(n \cdot \pi \cdot \frac{x}{L}) \cdot 
\frac{\sin(n \cdot \pi \cdot \frac{A}{L})}{n^4 \cdot \left[ 1 - \frac{\omega_0^2}{\omega^2} \right]}
\]

**For \( \omega_0 \to 0 \) the dynamic solution equals the "static" bending solution.**

3.3 Center Deflection

For the center deflection at x=L/2 we obtain:

\[
V\left(\frac{L}{2}, t\right) = \frac{2 \cdot F_0 \cdot L^3 \cdot \sin(\omega_0 \cdot t)}{\pi^4 \cdot E \cdot I} \times 
\sum_{k=0}^{\infty} \left[ (-1)^{k+1} \cdot \frac{\sin\left(\frac{2 \cdot k - 1}{2} \cdot \pi \cdot \frac{A}{L}\right)}{(2 \cdot k - 1)^4 \cdot \left[ 1 - \frac{\omega_0^2}{\omega_{2k-1}^2} \right]} \right]
\]
### 3.4 ASTM Specifications

The ASTM Specifications require that $A = \frac{L}{3}$ and $B = 2L/3$. This leads to the following possible $n$ numbers:

I. $n = 3, 6, 9, ...$ (Series $3k$; $k = 1, 2, 3, ...$);
II. $n = 1, 7, 13, ...$ (Series $(6k - 5)$; $k = 1, 2, 3, ...$);
III. $n = 5, 11, 17, ...$ (Series $(6k - 1)$; $k = 1, 2, 3, ...$).

The first series I is not relevant because in that case $\sin(3k \pi/3) = 0$. The general formula is:

$$V(x, t) = \frac{\sqrt{3} \cdot F_0 \cdot L^3 \cdot \sin(\omega_0 \cdot t)}{\pi^4 \cdot E \cdot I} \times \sum_{k=1}^{\infty} \left[ \frac{\sin \left( \frac{6k - 5}{L} \right) \cdot \frac{x}{L} \cdot \left( 1 - \frac{\omega_0^2}{\omega_{6k-1}^2} \right)}{(6k - 5)^4} \right]$$

The maximum deflection at $x = L/2$ is given by equation 108:

$$V_{\text{max}} = V \left( \frac{L}{2}, t \right) = \frac{\sqrt{3} \cdot F_0 \cdot L^3 \cdot \sin(\omega_0 \cdot t)}{\pi^4 \cdot E \cdot I} \times \sum_{k=1}^{\infty} \left[ \frac{(-1)^{k+1}}{(6k - 5)^4} \cdot \left( 1 - \frac{\omega_0^2}{\omega_{6k}^2} \right) \right]$$

The deflection at $x = L/3$ (the inner clamps; ASTM coordinate A) is given by:

$$V \left( \frac{L}{3}, t \right) = \frac{3 \cdot F_0 \cdot L^3 \cdot \sin(\omega_0 \cdot t)}{4 \cdot \pi^4 \cdot E \cdot I} \times \sum_{k=1}^{\infty} \left[ \frac{1}{(6k - 5)^4} \cdot \left( 1 - \frac{\omega_0^2}{\omega_{6k-1}^2} \right) \right] + \sum_{k=1}^{\infty} \left[ \frac{1}{(6k - 1)^4} \cdot \left( 1 - \frac{\omega_0^2}{\omega_{6k}^2} \right) \right]$$

37
3.5 First Order Approximation

For common values of the beam characteristics (E,b,h,L and I) and the frequencies $\omega_0$ it can be shown that the first term of the infinite series already accounts for more than 99 % of the total deflection (see figures 3.1 and 3.2). Therefore the deflection can be presented as:

$$V(x,t) = \frac{2 \cdot F_0 \cdot L^3 \cdot \sin \left( \frac{\pi}{L} \cdot x \right) \cdot \sin \left( \frac{\pi}{L} \cdot A \right)}{\pi^4 \cdot E \cdot I} \cdot \frac{\sin (\omega_0 \cdot L)}{1 - \frac{\omega_0^2}{\omega_i^2}} \quad 113$$

Furthermore it can be shown that the first term on the right hand side can be replaced by the static deflection equation without introducing errors. In fact the error with respect to the correct expression becomes less as will be shown later on in this paragraph.

$$\begin{aligned}
\text{Region I: } &0 \leq x \leq \frac{a}{2} \quad \Rightarrow \quad V(x,t) \approx \sin (\omega_0 \cdot L) \times \\
&\times \frac{F_0}{96 \cdot E \cdot I} \left[ + 12x^2 \left( 2L^2 + 2aL - a^2 \right) \right] \quad 114
\end{aligned}$$

$$\begin{aligned}
\text{Region II: } &\frac{a}{2} \leq x \leq \frac{L}{2} \quad \Rightarrow \quad V(x,t) \approx \sin (\omega_0 \cdot L) \times \\
&\times \frac{F_0}{96 \cdot E \cdot I} \left[ - 8x^3 + 12Lx^2 - 6a^2x + 3L.a^2 - 2L^3 \right] \quad 115
\end{aligned}$$

Using the coordinate system which is used in the dynamic case the formulas are:

$$\begin{aligned}
\text{Region I: } &A \leq x \leq \frac{L}{2} \quad \Rightarrow \quad V(x,t) \approx \sin (\omega_0 \cdot L) \times \\
&\times \frac{8 \cdot F_0 \cdot L^3 \cdot A}{96 \cdot E \cdot I \cdot L} \left[ 3 \cdot \frac{x}{L} - 3 \cdot \frac{A^2}{L^2} - \frac{A^2}{L^2} \right] \quad 116
\end{aligned}$$

$$\begin{aligned}
\text{Region II: } &0 \leq x \leq A \quad \Rightarrow \quad V(x,t) \approx \sin (\omega_0 \cdot L) \times \\
&\times \frac{8 \cdot F_0 \cdot L^3 \cdot x}{96 \cdot E \cdot I \cdot L} \left[ 3 \cdot \frac{A}{L} - 3 \cdot \frac{A^2}{L^2} - \frac{x^2}{L^2} \right]
\end{aligned}$$
To obtain an impression of the error introduced by the first order approximation, some calculations have been carried out. The following figures have been used:

<table>
<thead>
<tr>
<th>E [MPa]</th>
<th>h [m]</th>
<th>b [m]</th>
<th>L [m]</th>
<th>ρ [kg/m³]</th>
<th>F₀ [N]</th>
<th>ADWW [m]</th>
<th>ASTM [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>0.05</td>
<td>0.05</td>
<td>0.4</td>
<td>2300</td>
<td>1</td>
<td>0.135</td>
<td>0.1333</td>
</tr>
</tbody>
</table>

Firstly, we calculated the correct deflections (summing over 50 terms) at the center x=L/2 (Vc) for the DWW equipment (A/L = 135/400) and for the ASTM specification (A/L = 1/3). These deflections are given in figures 3.1 and 3.2 as a function of the frequency ω₀. We have also plotted in these figures the ratios (Rat1 : equation 109 and Rat1' : equation 111) of the first order approximations and the correct deflection Vc. As shown in these two figures the approximations based on equation 111 lead to better results for the given beam figures. A similar calculation has been performed for the deflection Va at the inner clamp (x=A ; A/L=135/400). The results are plotted in figure 3.3. Again the approximation based on equation 111 is much better in the frequency range of 10 - 100 Hz. Even as the equipment parameter A/L goes down to low values the approximation Rat1' based on equation 111 will give satisfying results for the center deflection Vc (figure 3.4 ; error < 1 %) and for the deflection Va at the inner clamp (figure 3.5 ; error < 3 %).

At one hand this is not really surprising, because for ω₀ → 0 the approximation ought to lead to the static solution. This is secured in the formulation of equation 111 using the static solution as a coefficient.

**DWW Specification: A/L = 135/400**

![Graph](https://example.com/graph.png)

**Figure 3.1** First order approximations for the deflection at x=L/2 as a function of frequency.
the frequency for A/L = 135/400 (DWW specifications); L = 0.4 m
ASTM Specification: \( A/L = 1/3 \)

![Graph](image1)

**Figure 3.2** First order approximations for the deflection at \( x=A \) as a function of the frequency for \( A/L = 1/3 \) (ASTM Specifications); \( L = 0.4 \text{ m} \)

DWW Specification: \( A/L = 135/400 \)

![Graph](image2)

**Figure 3.3** First order approximations for the deflection at \( x=A \) as a function of the frequency for \( A/L = 135/400 \) (DWW specifications); \( L = 0.4 \text{ m} \)
**Figure 3.4**  
First order approximations for the deflection at $x=L/2$ as a function of the frequency for $A/L = 1/10$; $L = 0.4$ m.

**Figure 3.5**  
First order approximations for the deflection at $x=A$ as a function of the frequency for $A/L = 1/10$; $L = 0.4$ m.
Based on these examples the following conclusion can be drawn:

1) For a frequency range of 0 to 100 Hz a first order approximation for the occurring deflections, which is based on equation 111, will lead to far better results (error < 1%) than the approximation which is based on equation 109.

2) Consequently it is recommended to use equation 111 for the back calculation of $E$ values from measurements. The error will be less than 1%.

3.6 Calculation of occurring strains in the beam

As already shown in chapter 2 for static bending, the first term of the solution, expressed in infinite series, will lead to unsatisfactory results for the calculation of the strains. In order to check this finding for dynamic bending, the following calculations have been carried out.

- Calculation of the exact strains at $X=L/2$ and $X=A$ for $A/L=135/400$ and $A/L=1/10$ as a function of the frequency $f_0$ (second derivation of equation 105 with a summation over 50 terms; figures 3.6 to 3.9).

- For $X=L/2$ (figures 3.6 and 3.8)
  - Calculation of the deflection $V_{\text{max}}$ at $X=L/2$ (equation 106 = measured value)
  - Adopting the first order approximation for the deflection (equation 110).
  - Calculation of the occurring strain according to this approximation and the ratio $Rat_1$ of this strain to the correct strain value (equation 114).

\[
\epsilon_{\text{back},1} = \frac{h}{2} \cdot \frac{\partial^2}{\partial x^2} V(x,t) = \frac{h}{2} \cdot \frac{\pi^2}{L^2} \cdot V_{\text{max}} ; \quad Rat_1 = \frac{\epsilon_{\text{back},1}}{\epsilon} \quad 117
\]

- Adopting the first order approximation for the deflection (equation 112).
- Calculation of the occurring strain according to this approximation and the ratio $Rat_1'$ of this strain to the correct strain value (equation 115).

\[
\epsilon_{\text{back},2} = \frac{h}{2} \cdot \frac{24}{3 \cdot L^2 \cdot 4 \cdot A^2} V_{\text{max}} ; \quad Rat_1' = \frac{\epsilon_{\text{back},2}}{\epsilon} \quad 118
\]

- Plotting of the two ratios $Rat_1$ and $Rat_1'$ and the correct strain as a function of the frequency $f_0$. 


For X=A (figures 3.7 and 3.9)

- Calculation of the deflection \( V_a \) at X=A (= measured value)
- Adopting the first order approximation for the deflection (equation 110).
- Calculation of the occurring strain at X=A according to this approximation and the ratio Rat1 of this strain to the correct strain value (equation 114).
- Calculating the (assumed) deflection \( V_{\text{max}1} \) at X=L/2
- Calculation of the occurring strain at X=L/2 according to this approximation and the ratio Rat2 of this strain to the correct strain value at X=L/2 (equation 116).

\[
V_{\text{max}1} = \frac{V_{\text{measured at } x=A}}{\sin(\pi \cdot \frac{A}{L})}
\]

\[
\varepsilon_{\text{back}1} = \frac{h}{2} \cdot \frac{\partial^2}{\partial x^2} V(x,t) = \frac{h}{2} \cdot \frac{\pi^2}{L^2} \cdot V_{\text{max}1} ; \quad \text{Rat}_1 = \frac{\varepsilon_{\text{back}1}}{\varepsilon}
\]

- Adopting the first order approximation for the deflection (equation 112).
- Calculating the (assumed) deflection \( V_{\text{max}2} \) at X=L/2
- Calculation of the occurring strain at X=L/2 according to this approximation and the ratio Rat2' of this strain to the correct strain value at X=L/2 (equation 117).

\[
V_{\text{max}2} = V_{\text{measured at } x=A} \cdot \frac{3 \cdot L^2 - 4 \cdot A^2}{12 \cdot A \cdot L - 16 \cdot A^2}
\]

\[
\varepsilon_{\text{back}2} = \frac{h}{2} \cdot \frac{\partial^2}{\partial x^2} V(x,t) = \frac{h}{2} \cdot \frac{24}{3 \cdot L^2 - 4 \cdot A^2} \cdot V_{\text{max}2} ; \quad \text{Rat}_2 = \frac{\varepsilon_{\text{back}2}}{\varepsilon}
\]

- Plotting of the three ratios Rat1, Rat2 and Rat2' and the correct strain as a function of the frequency \( f_o \).

As shown in the figures the calculation of the occurring strains based on equations 115 and 117 give better results (Rat1’ in figures 3.6 and 3.8 and Rat2' in figure 3.7 and 3.9).

It is advised to use equation 115 for the calculation of the maximum strain if the deflection is measured in the center (x=L/2) and to use equation 117 if the deflection is measured at the inner clamp (x=A).
Figure 3.6  Ratios of calculated strains and the theoretical strain in the center as a function of the frequency $\omega_0$. A/L=135/400; L=0.4 m; Deflection at x=L/2

Figure 3.7  The theoretical strains at x=A and x=L/2 and the ratios of calculated strains as a function of the frequency $\omega_0$. A/L=135/400; L=0.4 m; Deflection at x=A
Figure 3.8  Ratios of calculated strains and the theoretical strains in the center as a function of the frequency \( \omega \). \( A/L=1/10; L=0.4 \) m; Deflection measured at \( x=L/2 \).

Figure 3.9  Theoretical strains at \( x=A \) and \( x=L/2 \) and the ratios of calculated strains as a function of the frequency \( \omega_0 \). \( A/L=1/10; L=0.4 \) m; Deflection at \( x=A \).
4. Dynamic Bending of a Viscous-Elastic Beam (Ltot=L)

4.1. Homogenous differential equation (Q[x,t]=0)

Instead of the Young's modulus E the complex stiffness modulus $S_{mix}$ is introduced:

$$S_{mix} = \hat{S}_{mix} \cdot e^{i\phi} \quad (118)$$

First of all we consider the solution of the homogenous differential equation (Q[x,t]=0) in case of the free vibrations. Because of the viscous-elastic character of the material, the deflections will not last for ever but will disappear in time (damping).

So, the displacement $V(x,t)$ can be represented as:

$$V(x,t) = V(x) \cdot e^{i\omega_n t - \alpha_n t} \quad (119)$$

For the function $V(x)$ we adopt the infinite series in orthogonal sines. In this way the boundary conditions at x=0 and x=L are automatically fulfilled:

$$\hat{V}(x) = \sum_{n=1}^{\infty} D_n \cdot \sin \left( n \cdot \frac{\pi}{L} \cdot x \right) \quad (120)$$

Because the deflections at the two inner clamps (x=A and x=B=L-A) must be equal only odd numbers for $n$ are possible and those numbers for which $n \times A/L$ has an integer value. Invoking these equations into the differential equation leads to:

$$\hat{S}_{mix} \cdot I \cdot \frac{\partial^4}{\partial x^4} V(x,t) - \rho \cdot b \cdot h \cdot \frac{\partial^2}{\partial t^2} V(x,t) = Q(x,t) \quad (121)$$

$$\sum_{n=1}^{\infty} D_n \cdot \hat{V}(x) \cdot \left[ \hat{S}_{mix} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} \cdot e^{i\phi} + \rho \cdot b \cdot h \cdot \left[ i \cdot \omega_n - \alpha_n \right]^2 \right] = 0 \quad (122)$$

This relation has to hold for each x value. Thus, the eigen frequency $\omega_n$ and the damping $\alpha_n$ are determined by the following two equations:

$$\hat{S}_{mix} \cdot I \cdot \cos(\phi) \cdot \frac{n^4 \cdot \pi^4}{L^4} = \rho \cdot b \cdot h \cdot \left[ \omega_n^2 - \alpha_n^2 \right] \quad (123)$$

$$\hat{S}_{mix} \cdot I \cdot \sin(\phi) \cdot \frac{n^4 \cdot \pi^4}{L^4} = \rho \cdot b \cdot h \cdot 2 \cdot \omega_n \cdot \alpha_n \quad (124)$$

Elimination of the phase lag $\phi$ leads to:

$$\hat{S}_{mix} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} = \rho \cdot b \cdot h \cdot \left[ \omega_n^2 + \alpha_n^2 \right] \quad (125)$$

So, the eigen frequencies $\omega_n$ are given by:

$$\omega_n = n^2 \cdot \pi^2 \cdot \sqrt{\frac{\hat{S}_{mix} \cdot I}{M_{beam} \cdot L^3}} \cdot \sqrt{\frac{\cos(\phi) + 1}{2}} = n^2 \cdot \pi^2 \cdot \sqrt{\frac{\hat{S}_{mix} \cdot I}{M_{beam} \cdot L^3}} \cdot \cos(\phi) \quad (126)$$
and the damping ratios $\alpha_n$ are given by:

$$\alpha_n = n^2 \cdot \pi^2 \cdot \sqrt{\frac{\hat{S}_{\text{mix}} \cdot I}{M_{\text{beam}} \cdot L^3}} \cdot \sin \left( \frac{\varphi}{2} \right) \quad (127)$$

If we are interested in solutions of the homogenous differential equation with a prefixed frequency $\omega_0$, we have two types of solutions. The first one contains a decreasing term with the time $t$. This type of solution is needed if after a while the driving force $F_n \cdot e^{i\omega_0 t}$ disappears and the deflection will diminish in time:

$$V(x, t) = \left[ C_1 \cdot e^{\beta x} + C_2 \cdot e^{-\beta x} + C_3 \cdot e^{i \beta x} + C_4 \cdot e^{-i \beta x} \right] \cdot e^{i \omega_0 t} \cdot e^{\alpha_i t} \quad (128)$$

in which the parameters $\beta$ and $\omega_0^*$ are given by:

$$\beta = 4 \frac{2 \cdot \rho \cdot b \cdot h \cdot \omega_0^2}{\sqrt{\hat{S}_{\text{mix}} \cdot I \cdot [\cos(\varphi) + 1]}} \quad \text{and} \quad \alpha_i^* = \omega_0^* \cdot \tan \left[ \frac{\varphi}{2} \right] \quad (129)$$

Problems which need this type of solution are not discussed in this report.

The second type of solution with a prefixed frequency $\omega_0$ is more interesting:

$$V(x, t) = C_n \cdot e^{(\beta_i e^{i \alpha_0 t}) x} \cdot e^{i \omega_0 t} \quad (130)$$

This will lead to the following equation:

$$\hat{S}_{\text{mix}} \cdot I \cdot \beta_i^4 \cdot e^{i(4 \alpha_0)} - \rho \cdot b \cdot h \cdot \omega_0^2 = 0 \quad (131)$$

From which it follows that $\alpha_0$ and $\beta_i$ are given by:

$$\begin{align*}
\sin(\varphi - 4 \cdot \alpha_0) &= 0 \quad \rightarrow \alpha_0 = \frac{\varphi}{4} + \frac{k}{4} \cdot \pi \\
\beta_i^4 &= \left[ \frac{\rho \cdot b \cdot h \cdot \omega_0^2}{\hat{S}_{\text{mix}} \cdot I} \right]^2
\end{align*} \quad (130)$$

These solutions are needed in case the length $L_{\text{tot}}$ is longer than the distance $L$. This problem is discussed in chapter 7. The equivalent elastic case ($\varphi = 0$) is dealt with in chapter 6.

It should be marked once again that the solutions of the homogenous differential equation (with a prefixed frequency $\omega_0$) are not needed in the case that the length $L_{\text{tot}}$ equals the distance $L$ between the two outer clamps ($\Delta = 0$). This case ($\Delta = 0$) is discussed in the following
paragraphs.
4.2 Force controlled bending

First of all we consider the case in which the forces (point loads at x=A and x=B; \( F_A = F_B = F_0 / 2 \times e^{i\omega t} \)) at the two inner clamps are given. We call it "Force controlled bending" because the forces at the two inner clamps are given and the deflection of the beam has to be determined. In order to account for the viscous-elastic character of asphalt the following procedure is used. The forces \( F_A(t) \) and \( F_B(t) \) are transformed into an equal force distribution \( Q(x,t) \) along the beam. \( Q(x,t) \) will be presented in an infinite series of orthogonal sines as usual.

The two inner clamps are symmetrically placed around the center. So, the force amplitude \( F_0 \) is divided equally over the two clamps:

\[
F( x , t ) = \left[ \frac{F_0}{2} \cdot \delta(x = A) + \frac{F_0}{2} \cdot \delta(x = B) \right] e^{i\omega t} = \hat{Q}( x ) e^{i\omega t} \tag{131}
\]

\[
\hat{Q}( x ) = \frac{2 \cdot F_0}{L} \sum_{n=1,3,5}^\infty \sin\left( n \cdot \frac{\pi}{L} \cdot x \right) = \frac{2 \cdot F_0}{L} \sum_{n=1,3,5}^\infty P_n(x) \tag{134}
\]

We assume that the deflection can be given by:

\[
V( x , t ) = e^{i(\omega_0 t + \phi_n^*)} \cdot C_n \cdot \sin\left( n \cdot \frac{\pi}{L} \cdot x \right) \tag{135}
\]

This type of solution meets the requirements for the boundary conditions at x=0 and x=L. Invoking these equations in the 4th order differential equation lead to:

\[
\left\{ \sum_{n=1,3}^\infty \left[ C_n \cdot P_n(x) \cdot \left[ \hat{S}_{\text{mix}} \cdot I \cdot e^{i\phi^*} \cdot \frac{n^4 \cdot \pi^4}{L^4} - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot e^{-i\phi_n} \right] \right] = \frac{2 \cdot F_0}{L} \sum_{n=1,3,5}^\infty P_n(x) \right\} \tag{136}
\]

This has to be true for all x values. Therefore we obtain the two following equations:

\[
C_n \cdot \left[ \hat{S}_{\text{mix}} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} \cdot \cos\left( \phi - \phi_n^* \right) - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot \cos\left( \phi_n^* \right) \right] = \frac{2 \cdot F_0}{L} \tag{13237}
\]

\[
C_n \cdot \left[ \hat{S}_{\text{mix}} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} \cdot \sin\left( \phi - \phi_n^* \right) + \rho \cdot b \cdot h \cdot \omega_0^2 \cdot \sin\left( \phi_n^* \right) \right] = 0 \tag{133}
\]

We will introduce the parameter \( \xi_n \), comparable to the eigen frequency \( \omega_0 \) in the elastic case:

\[
\xi_n^2 = \frac{n^4 \cdot \pi^4}{L^4} \cdot \frac{\hat{S}_{\text{mix}} \cdot I}{\rho \cdot b \cdot h} \tag{134}
\]

The phase lag \( \phi_n^* \) is given by:

\[
\frac{\sin\left( \phi_n^* \right)}{\cos\left( \phi_n^* \right)} = \tan\left( \phi_n^* \right) = \frac{\sin\left( \phi \right)}{\cos\left( \phi \right) - \omega_0^2 / \xi_n^2} = Z_1 \tag{135}
\]
The coefficient $C_n$ can be determined by:

$$C_n^2 \cdot \xi_n^4 = \frac{4 \cdot F_0^2}{L^2 \cdot \rho^2 \cdot b^2 \cdot h^2} + \frac{4 \cdot F_0 \cdot \omega_0^2}{L \cdot \rho \cdot b \cdot h} \cdot C_n \cdot \cos (\varphi_n^*) + C_n^2 \cdot \omega_0^4$$  \hspace{1cm} (136)

$$\Rightarrow \cos (\varphi_n^*) = \frac{C_n^2 \cdot \left[ \xi_n^4 - \omega_0^4 \right] - \frac{4 \cdot F_0^2}{M_{\text{beam}}}}{\frac{4 \cdot F_0}{M_{\text{beam}}} \cdot C_n \cdot \omega_0^4} = Z_n^2$$  \hspace{1cm} (137)

Substitution of $\cos(\varphi_n^*)$ and $\sin(\varphi_n^*)$ leads finally to:

$$C_n = \frac{2 \cdot F_0 \cdot L^3}{\pi \cdot \dot{S}_{\text{mix}} \cdot I \cdot n^4} \cdot \frac{1}{\sqrt{1 - 2 \cdot \frac{\omega_0^2}{\xi_n^2} \cdot \cos (\varphi) + \frac{\omega_0^4}{\xi_n^4}}}$$  \hspace{1cm} (138)

$$\cos (\varphi_n^*) = \frac{\cos (\varphi) \cdot \omega_0^2}{\xi_n^2}$$  \hspace{1cm} (139)

The following coefficients are handy in further calculations:

$$C_n \cdot \cos (\varphi_n^*) = \frac{2 \cdot F_0}{\rho \cdot b \cdot h \cdot L \cdot \xi_n^2} \cdot \frac{\cos (\varphi) - \frac{\omega_0^2}{\xi_n^2}}{1 - 2 \cdot \frac{\omega_0^2}{\xi_n^2} \cdot \cos (\varphi) + \frac{\omega_0^4}{\xi_n^4}}$$  \hspace{1cm} (140)

$$C_n \cdot \sin (\varphi_n^*) = \frac{2 \cdot F_0}{\rho \cdot b \cdot h \cdot L \cdot \xi_n^2} \cdot \frac{\sin (\varphi)}{1 - 2 \cdot \frac{\omega_0^2}{\xi_n^2} \cdot \cos (\varphi) + \frac{\omega_0^4}{\xi_n^4}}$$  \hspace{1cm} (141)

The deflection $V(x,t)$ along the beam can now be represented by:
\[ V( x , t ) = e^{i \omega t} \cdot \sum_{n=1,3}^{\infty} C_n \cdot P_n(x) \cdot e^{-i \phi_n} = \]

\[ = \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot \hat{S}_{mix} \cdot I} \cdot e^{i \omega t} \cdot \sum_{n=1,3}^{\infty} \left[ \sin \left( n \cdot \frac{\pi}{L} \right) \cdot \sin \left( n \cdot \frac{x}{L} \right) \cdot e^{i \cdot \arctan \left[ \frac{\xi_n^2 \cdot \sin(\phi)}{\xi_n \cdot \cos(\phi)} \right]} \right] \]

\[ \sqrt{n^4 \cdot \left( 1 - 2 \cdot \cos(\phi) \cdot \frac{\omega_0}{\xi_n^2} + \frac{\omega_0^4}{\xi_n^4} \right)} \]

Or by:
\[ V(x,t) = \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot \hat{S}_{mix} \cdot I} \cdot e^{i \omega t} \cdot \sum_{n=1,3,5}^{\infty} \left[ \frac{P_n(x) \cdot e^{-i \phi_n}}{n^4 \cdot \sqrt{1 - 2 \cdot \cos(\phi) \cdot \frac{\omega_0}{\xi_n^2} + \frac{\omega_0^4}{\xi_n^4}}} \right] \]

(142)

Notice that, for each component in the series, the phase lag between the deflection component \( V_n(x,t) \) and the ‘force’ \( Q(x,t) \) or \( F(t) \) is a constant, which only depends on \( n \) and not on \( x \). However, the total phase lag \( \phi^* \) between the total deflection \( V(x) \) at a point \( x \) and the force \( F(t) \) will depend on the distance \( x \), as will be shown.

Because \( \phi \) will be positive for viscous-elastic materials \((0<\phi<90^o)\) and so \( \sin(\phi) \) will be positive, it follows from equation 146 that the sinus of the phase lag \( \phi_n \) will always be positive. Therefore the phase lag \( \phi_n \) will be between 0 and 180\(^o\). This remark is relevant in computer calculations if the arctangent function is used, which is defined for \(-\pi/2<\phi<\pi/2\). See also chapter 8.

The amplitude of the (measured) deflection at a point \( x \) is given by:

\[ \tan(\phi^*) = \frac{\sum_{n=1,3,5}^{\infty} \left[ P_n(x) \cdot \sin(\phi_n^*) \right]}{\sqrt{n^4 \cdot \left( 1 - 2 \cdot \cos(\phi) \cdot \frac{\omega_0}{\xi_n^2} + \frac{\omega_0^4}{\xi_n^4} \right)}} \]

(144)

The phase lag \( \phi^* \) for the deflection \( V(x,t) \) at a point \( x \) is given by:
So, the total phase lag $\varphi^*$ between the deflection $V(x,t)$ at a point $x$

and the force $F(t)$ at the inner clamps depends on the distance $x$.
4.3. Prescribed Displacement Profile without phase lag differences

In this case we require that there will be no differences in phase lags between the deflections along the beam. Instead of the force distribution \( Q(x,t) \) the deflection \( V(x,t) \) is given along the beam and we want to determine the required force distribution \( Q(x,t) \) to achieve such a displacement profile. One could call this "Displacement Controlled Bending". However, as will be shown the required force distribution \( Q(x,t) \) can not be achieved by point loads at the two inner clamps. The deflection will be given by:

\[
V(\ x, \ t) = \hat{V}_0(\ x) \cdot e^{i\omega_0 t}
\]  

(147)

The amplitude of the deflection at each point \( x \) can be described by a series of orthogonal sine’s leading to:

\[
\hat{V}_0(x) = \sum_{n=1,3,5}^\infty C_n \cdot \sin\left( n \cdot \pi \cdot \frac{A}{L} \right) \cdot \sin\left( n \cdot \pi \cdot \frac{x}{L} \right) = \sum_{n=1,3,5}^\infty C_n \cdot P_n(x)
\]

(148)

Because of the symmetry around \( x=L/2 \) only odd numbers for \( n \) have to be taken into account. The coefficients \( C_n \) are determined by:

\[
C_n \cdot \sin\left( n \cdot \pi \cdot \frac{A}{L} \right) = \frac{2}{L} \int_0^L \hat{V}_0(x) \cdot \sin\left( n \cdot \pi \cdot \frac{x}{L} \right) \, dx
\]

(149)

The problem is now reversed compared to the force controlled bending case. Here we have to determine the force distribution \( Q(x,t) \) along the beam, which will cause such a deflection profile. The required force distribution can be presented by:

\[
Q(\ x, \ t) = \sum_{n=1,3,5}^\infty e^{i(\omega_n \cdot t + \phi_n^\circ)} \cdot \frac{2 \cdot Q_n}{L} \cdot P_n(x)
\]

(150)

So, there is a difference in phase lag between \( Q(x,t) \) and \( V(x,t) \), which will depend on the coordinate \( x \) along the beam.

Invoking the relations into the fourth order differential equation leads to two equations (for each component of the series):

\[
\frac{2 \cdot Q_n}{L} \cdot \cos(\phi_n^\circ) = \frac{n^4 \cdot \pi^4 \cdot \hat{S}_{\text{mix}} \cdot I}{L^4} \cdot \left[ \cos(\phi) \cdot \frac{\omega_0^2}{\xi_n^2} \right] \cdot C_n
\]

(151)

And

\[
\frac{2 \cdot Q_n}{L} \cdot \sin(\phi_n^\circ) = \frac{n^4 \cdot \pi^4 \cdot \hat{S}_{\text{mix}} \cdot I}{L^4} \cdot \sin(\phi) \cdot C_n
\]

(152)

As might be expected the phase lag for each component will be:

\[
\frac{\sin(\phi_n^\circ)}{\cos(\phi_n^\circ)} = \frac{\xi_n^2 \cdot \sin(\phi)}{\xi_n^2 \cdot \cos(\phi) \cdot \omega_0^2} = Z_1
\]

(153)

The coefficient \( Q_n \) is given by:
\[
Q_n = \frac{C_n \cdot \pi^4 \cdot \hat{S}_{\text{mix}} \cdot I \cdot n^4}{2 \cdot L^3} \cdot \sqrt{1 - 2 \cdot \frac{\omega_0^2}{\xi_n^2} \cdot \cos (\varphi) + \frac{\omega_0^4}{\xi_n^4}} \quad (154)
\]

The force distribution \(Q(x,t)\) is determined by:

\[
Q(x,t) = \hat{Q}(x) \cdot e^{i(\omega t + \varphi)} = \frac{\pi^4 \cdot \hat{S}_{\text{mix}} \cdot I}{L^4} \sum_{n=1,3,5} C_n \cdot P_n(x) \cdot e^{i(\omega t + \varphi)} \cdot n^4 \cdot \sqrt{1 - 2 \cos (\varphi) \cdot \frac{\omega_0^2}{\xi_n^2} + \frac{\omega_0^4}{\xi_n^4}} \quad (155)
\]

5) The amplitude and the phase lag of the force distribution (along the beam) will be given by:

\[
\hat{Q}(x) = \frac{\pi^4 \cdot \hat{S}_{\text{mix}} \cdot I}{L^4} \times \left( \sum_{n=1,3,5} P_n(x) \cdot C_n \cdot n^4 \cdot \left[ \cos (\varphi) - \frac{\omega_0^2}{\xi_n^2} \right] \right)^2 + \left( \sum_{n=1,3,5} P_n(x) \cdot C_n \cdot n^4 \cdot \sin (\varphi) \right)^2 \quad (156)
\]

and

\[
\tan (\varphi^*) = \frac{\sum_{n=1,3,5} P_n(x) \cdot C_n \cdot n^4 \cdot \sin (\varphi)}{\sum_{n=1,3,5} P_n(x) \cdot C_n \cdot n^4 \cdot \cos (\varphi) - \frac{\omega_0^2}{\xi_n^2}} \quad (157)
\]

As indicated by the equation above the phase lag in the force distribution \(Q\) along the beam depends on the coordinate \(x\). Therefore such a force distribution cannot be achieved by simple point load forces at the two inner clamps.

\textit{A prescribed deflection profile along the beam with a constant phase lag between the deflections and the force distribution cannot be simulated by point load forces.}
4.4 First order approximation for the "Force controlled bending" and back calculation of the stiffness modulus from measurements

The distribution of the deflection along the beam V(x,t) is given by equation 147 (or 148), the amplitude at each point by equation 149 and the phase lag \( \varphi^*(x) \) by equation 150 (or 151). If we only take the first term of the series, we will obtain a first order approximation:

\[
\hat{V}(x) = 2 \cdot F_0 \cdot \frac{L^3}{\pi^4 \cdot S_{\text{mix}} \cdot I} \cdot \frac{P_1(x)}{\sqrt{1 - 2 \cdot \cos(\varphi) - \frac{\omega_0^2}{\xi_1^2} + \frac{\omega_0^4}{\xi_1^4}}}
\]

(158)

\[
\tan(\varphi^*) = \frac{\sin(\varphi)}{\cos(\varphi) - \frac{\omega_0^2}{\xi_1^2}}
\]

(159)

Notice that these expressions are equal in form to the solution of a single viscous-elastic spring-mass system with a mass \( M_{\text{equiv.}} \) and a visco-elastic spring constant \( K \times e^{i\varphi} \).

The solution for a single viscous-elastic spring-mass system is:

\[
\hat{V}(x) \cdot \left[ K \cdot e^{i(\varphi - \varphi^*_1)} - M_{\text{equiv.}} \cdot \omega_0^2 \cdot e^{i\varphi^*_1} \right] = F_0
\]

(160)

this leads to two equations:

\[
\hat{V}(x) \cdot \left[ K \cdot \cos(\varphi - \varphi^*_1) - M_{\text{equiv.}} \cdot \omega_0^2 \cdot \cos(\varphi^*_1) \right] = F_0
\]

(161)

\[\hat{V}(x) \cdot \left[ K \cdot \sin(\varphi - \varphi^*_1) + M_{\text{equiv.}} \cdot \omega_0^2 \cdot \sin(\varphi^*_1) \right] = 0
\]

(167)

Elimination of the phase lag \( \varphi^*_1 \) leads to:

\[
\hat{V}(x) = \frac{F_0}{K} \cdot \frac{1}{\sqrt{1 - 2 \cdot \cos(\varphi) - \frac{M_{\text{equiv.}}}{K} \cdot \omega_0^2 + \frac{M_{\text{equiv.}}}{K^2} \cdot \omega_0^4}}
\]

(162)

If, for the interval \( A < x < L/2 \), the function \( 2P_1(x)/\pi^4 \) in equation 163 is replaced by the equivalent static function, one will obtain a better approximation for the frequency range 0 - 100 Hz, as already is shown in paragraph 3.5. The replacement is given by:

\[
\frac{1}{R} = \frac{2}{\pi^4} \cdot P_1(x) \Rightarrow \frac{1}{R} = \frac{1}{12 \cdot L} \cdot \left[ +3 \cdot \frac{x}{L} - 3 \cdot \frac{x^2}{L^2} - \frac{A^2}{L^2} \right]
\]

(163)

Notice that this replacement is also relevant for the determination of the equivalent mass.
The parameter $\xi_1$ is still defined as:

$$\xi_1^2 = \frac{\pi^4 \cdot S_{\text{mix}} \cdot I}{L^3 \cdot M_{\text{beam}}} = \frac{K}{M_{\text{equiv.}}} \quad (164)$$

Without the replacement the equivalent mass will be given by:

$$M_{\text{equiv.}} = \frac{I}{\pi^4} \cdot \frac{\pi^4}{2 \cdot \sin (\pi \frac{A}{L}) \cdot \sin (\pi \frac{x}{L})} \cdot M_{\text{beam}} = \frac{R}{\pi^4} \cdot M_{\text{beam}} \quad (165)$$

With the partial replacement of the coefficient the expression for the equivalent mass becomes:

$$M'_{\text{equiv.}} = \frac{I}{\pi^4} \cdot \frac{1}{A \cdot \frac{L}{L}} \cdot \left[ 3 \cdot \frac{x}{L} - 3 \cdot \frac{x^2}{L^2} - \frac{A^2}{L^2} \right] \cdot M_{\text{beam}} = \frac{R'}{\pi^4} \cdot M_{\text{beam}} \quad (166)$$

For $x = L/2$, $A = 0.135$ m and $L = 0.400$ m the differences between $M_{\text{equiv.}}$ and $M'_{\text{equiv.}}$ is small: $M_{\text{equiv.}} = 0.5731 \ M_{\text{beam}}$ and $M'_{\text{equiv.}} = 0.5738 \ M_{\text{beam}}$.

The straight forward first order approximation is characterized by:

$$K = R \cdot \frac{I}{L^3} \cdot S_{\text{mix}} \ ; \ M_{\text{equiv.}} = \frac{R}{\pi^4} \cdot M_{\text{beam}} \quad (167)$$

The modified first order approximation is characterized by:

$$K' = R' \cdot \frac{I}{L^3} \cdot S_{\text{mix}} \ ; \ M'_{\text{equiv.}} = \frac{R'}{\pi^4} \cdot M_{\text{beam}} \quad (168)$$

As shown in figure 4.1 the modified first order approximation is only slightly better for the calculation of the deflection.
Deflection Ratios in Centre
Beam Length 400 mm

![Deflection Ratios Graph](image)

**Figure 4.1** Deflection ratios as a function of the frequency; $S_{\text{mix}}=2000$ MPa; $\varphi=20^\circ$
Full line: $1^{\text{st}}$ order ; Dotted line: Modified $1^{\text{st}}$ order approximation

*Strain Calculation*

The exact occurring strain amplitude in the edge of the beam has to be calculated from equation 147 in the following way:

$$\hat{\varepsilon}(x) = \frac{h}{2} \left| \frac{\partial^2}{\partial x^2} V(x,t) \right|$$

(169)

We denote the straightforward first order approximation for the deflection by $V_1(x)$, equation 163 or 168, and the modified first order approximation by $V_{1/}(x)$, equation 168 using $K'$ and $M_{\text{equiv}}$.

Using these approximations the amplitudes of the calculated strains are given by:

$$\varepsilon_1(x) = \frac{h}{2} \frac{\pi^2}{L^2} V_1(x)$$

(170)

$$\varepsilon_{1/}(x) = \frac{3}{3 \cdot x \cdot L - 3 \cdot x^2 - A} \cdot V_{1/}(x)$$

As shown in figure 4.2 the modified approximation will lead to a better result in the calculation of the strain amplitude for the frequency range from 0-100 Hz.

Overall the modified first order approximation is better for the calculation of both the deflection and the strain. Therefore this approximation will be preferable for the back calculation procedure.
Figure 4.2  Strain ratios as a function of the frequency; $S_{mix}=2000$ MPa; $\varphi=20^\circ$
Full line: 1st order; Dotted line: Modified 1st order approximation

**Back calculation**

For the back calculation of the stiffness modulus $S_{mix}$ and the phase lag $\varphi$ from the measured deflection $|V(x)|$ and the phase lag $\varphi'(x)$, we use the first order approximation in a reverse way.

The back calculated stiffness modulus is given by:

$$
S_{mix,\text{backc.}} = \frac{F_0 \cdot L^3}{V(x) \cdot R \cdot I} \sqrt{1 + 2 \cdot \cos (\varphi^*(x)) \cdot M_{\text{equiv.}} \cdot \frac{\dot{V}(x)}{F_0} \cdot \omega^2 + \left[ M_{\text{equiv.}} \cdot \frac{\dot{V}(x)}{F_0} \cdot \omega^2 \right]^2}
$$

(171)

The back calculated phase lag is given by:

$$
tan (\varphi_{\text{backc.}}) = \frac{\sin (\varphi^*(x))}{\cos (\varphi^*(x)) + M_{\text{equiv.}} \cdot \frac{\dot{V}(x)}{F_0} \cdot \omega^2}
$$

(172)

In case of the modified approximation, the parameters $R$ and $M_{\text{equiv.}}$ are replaced by $R'$ and $M'_{\text{equiv.}}$.

In figure 4.3 the ratios of the back calculated $S_{mix}$ and the input value are given as a function of the frequency for the straight forward 1st order approximation and the modified 1st order approximation.
Figure 4.3  Ratio of back calculated $S_{\text{mix}}$ values as a function of the frequency; Full line: 1st order; Dotted line: Modified 1st order approximation.

In figure 4.4 this is done for the difference in the back calculated phase lag and the input value. As expected the modified approximation turns out to be only slightly better in the frequency range from 0-100 Hz.
Figure 4.4  Difference in back calculated phase lag as a function of the frequency; Full line: 1st order; Dotted line: Modified 1st order approximation.
5. Concentrated masses at the two inner clamps

5.1 Development of the mass force into a force distribution

In practice the force \( F(t) \) will be measured by a sensor above the clamps. Therefore the actual forces on the beam at the two clamps are lower by the movements of the masses of the clamps etc. The total extra moving mass is defined by the parameter \( M_{\text{clamp}} \). The "point load forces" \( F_{\text{mass}}(t) \) related to this moving mass is given by:

\[
F_{\text{mass}}(x, t) = \frac{M_{\text{clamp}}}{2} \left[ \frac{\partial^2}{\partial t^2} V(x, t)|_{x=A} \cdot \delta(x = A) + \frac{\partial^2}{\partial t^2} V(x, t)|_{x=B} \cdot \delta(x = B) \right]
\]

Assuming the same type of solution for the deflection \( V(x, t) \) as if there were no concentrated masses, this equation can be rewritten as:

\[
F_{\text{mass}}(x, t) = -\omega_0^2 e^{i\omega \tau} \frac{M_{\text{clamp}}}{2} \left[ \hat{V}(A) e^{-i\phi^*(A)} \cdot \delta(x = A) + \hat{V}(B) e^{-i\phi^*(B)} \cdot \delta(x = B) \right]
\]

in which \( V(A), V(B), \phi^*(A) \) and \( \phi^*(B) \) are yet unknown. This force can be transformed into an equal force distribution \( Q_{\text{mass}}(x, t) \) along the beam. \( Q_{\text{mass}}(x, t) \) is given in an infinite series of orthogonal sines. The two inner clamps are symmetrically placed around \( x=L/2 \), leading to:

\[
Q_{\text{mass}}(x, t) = \frac{2M_{\text{clamp}}}{L} \hat{V}(A) \omega_0^2 e^{i(\omega \xi_n + \phi^*(A))} \sum_{n=1,3,5}^\infty P_n(x)
\]

with \( P_n(x) = \sin\left(n \cdot \pi \cdot \frac{A}{L}\right) \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L}\right) \)

Invoking the relation into the differential equation 94 leads to:

\[
C_n \left[ \tilde{S}_{\text{mix}} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} \cdot \cos(\phi - \phi_n^*) - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot \cos(\phi_n^*) \right] = \frac{2}{L} F_\theta + \frac{2M_{\text{clamp}}}{L} \omega_0^2 \hat{V}(A) \cdot \cos(\phi^* [A])
\]

and

\[
C_n \left[ \tilde{S}_{\text{mix}} \cdot I \cdot \frac{n^4 \cdot \pi^4}{L^4} \cdot \sin(\phi - \phi_n^*) + \rho \cdot b \cdot h \cdot \omega_0^2 \cdot \sin(\phi_n^*) \right] = -\frac{2}{L} M_{\text{clamp}} \omega_0^2 \hat{V}(A) \cdot \sin(\phi^* [A])
\]

The procedure will be to solve both \( C_n \) and \( \phi_n^* \) as a function of \( \xi_n \) and \( \phi^*[A] \). The second step is to determine the amplitude and total phase lag at \( x=A \) which must equal \( V[A] \) and \( \phi^*[A] \). However, this will lead to complicated equations which have to be solved in a numerical way.

Introducing the parameter \( \xi_n \) and \( M_{\text{beam}} \).
\[ \varepsilon_n^2 = \frac{h^4 \cdot \pi^4}{L^4} \cdot \frac{S_{\text{mix}} \cdot I}{\rho \cdot b \cdot h} \quad \text{and} \quad M_{\text{beam}} = \rho \cdot b \cdot h \cdot L \]  

(185)

Starting with:

\[ C_n \cdot \left[ \xi_n^2 \cdot \cos \left( \varphi - \varphi_n^* \right) - \omega_n^2 \cdot \cos \left( \varphi_n^* \right) \right] = E_1 + E_2 \cdot \cos \left( \varphi^* [A] \right) \]  

(186)

\[ \begin{aligned}
& \left\{ \begin{aligned}
C_n \cdot \left[ \xi_n^2 \cdot \sin \left( \varphi - \varphi_n^* \right) + \omega_n^2 \cdot \sin \left( \varphi_n^* \right) \right] = - E_2 \cdot \sin \left( \varphi^* [A] \right) \\
& \quad \text{with } E_1 = \frac{2 \cdot F_0}{M_{\text{beam}}} \quad \text{and} \quad E_2 = \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot \omega_n^2 \cdot \hat{\nu} (A)
\end{aligned} \right.
\]  

(187)

\[ C_n \cdot \left[ \xi_n^2 \cdot \cos \left( \varphi \right) - \omega_n^2 \right] = E_1 \cdot \cos \left( \varphi^* [A] \right) + E_2 \cdot \cos \left( \varphi^* [A] - \varphi_n^* \right) \]  

(188)

\[ C_n \cdot \left[ \xi_n^2 \cdot \sin \left( \varphi \right) \right] = E_1 \cdot \sin \left( \varphi^* [A] \right) - E_2 \cdot \sin \left( \varphi^* [A] - \varphi_n^* \right) \]  

(189)

\[ C_n^2 \cdot \left[ \xi_n^2 + \omega_n^4 - 2 \cdot \xi_n^2 \cdot \omega_n^2 \cdot \cos \left( \varphi \right) \right] = E_1^2 + E_2^2 + 2 \cdot E_1 \cdot E_2 \cdot \cos \left( \varphi^* [A] \right) \]  

(190)

\[ \begin{aligned}
& \left\{ \begin{aligned}
C_n \cdot \left[ D_1 \cdot \cos \left( \varphi_n^* \right) + D_2 \cdot \sin \left( \varphi_n^* \right) \right] = E_1 + E_2 \cdot \cos \left( \varphi^* [A] \right) \\
& \quad \text{with } D_1 = \xi_n^2 \cdot \cos \left( \varphi \right) - \omega_n^2 \quad \text{and} \quad D_2 = \xi_n^2 \cdot \sin \left( \varphi \right)
\end{aligned} \right.
\]  

(191)

One will obtain:

\[ C_n \cdot \sin \left( \varphi_n^* \right) = \frac{E_1 + E_2 \cdot \cos \left( \varphi^* [A] \right)}{D_1^2 + D_2^2} \cdot D_2 + E_2 \cdot \sin \left( \varphi^* [A] \right) \cdot D_1 \]  

(192)

\[ C_n \cdot \cos \left( \varphi_n^* \right) = \frac{E_1 + E_2 \cdot \cos \left( \varphi^* [A] \right)}{D_1^2 + D_2^2} \cdot D_2 - E_2 \cdot \sin \left( \varphi^* [A] \right) \cdot D_1 \]  

(193)

The last two equations give the coefficients \( C_n \) and \( \varphi_n^* \) as a function of the amplitude and phase lag at \( x=A \) ( \( V[A] \) and \( \varphi^*[A] \) ). The amplitude and phase lag at \( x=A \) is given by the following two equations.

\[ \hat{V}^2 [A] = \left( \sum \left[ C_n \cdot \cos \left( \varphi_n^* \right) \cdot P_n [A] \right] \right)^2 + \left( \sum \left[ C_n \cdot \sin \left( \varphi_n^* \right) \cdot P_n [A] \right] \right)^2 \]  

(194)

\[ \tan \left( \varphi^*[A] \right) = \frac{\sum \left[ C_n \cdot \sin \left( \varphi_n^* \right) \cdot P_n [A] \right]}{\sum \left[ C_n \cdot \cos \left( \varphi_n^* \right) \cdot P_n [A] \right]} \]  

(195)

Solving these two equations will give the amplitude \( V[A] \) and \( \varphi^*[A] \) as a function of the (material) phase lag \( \varphi \) and stiffness modulus \( S_{\text{mix}} \).
5.2 Simplified procedure for the deflection calculation.

The procedure, outlined in paragraph 5.1, is quite complex. However, a rather good approximation can be obtained in a simple way.

The only condition is that the phase lag does not vary (much) along the beam.

The deflection at the clamp $V(A,t)$ is:

$$V(A,t) = \sum_{l=3}^{\infty} \left[ C_n \cdot P_n(A) \cdot e^{i(\alpha_l t - \varphi_n^*)} \right]$$

A good approximation is already given by the first term:

$$V(A,t) \approx g_0 \cdot C_1 \cdot P_1(A) \cdot e^{i(\alpha_1 t - \varphi_1^*)}; \text{with } g_0 = 1$$

The amplitude is given by:

$$\dot{V}(A) = g_0 \cdot C_1 \cdot P_1(A)$$

The amplitude can also be written as:

$$\dot{V}(A) = g_0 \cdot f_n \cdot C_n \cdot P_n(A)$$

The parameter $f_n$, the ratio of the coefficient $C_1$ and $C_n$, is equal to:

$$f_n = \frac{C_1}{C_n}; \text{with } g_n \text{ close to } 1$$

The expression for the force $Q_{mass}(x,t)$ can now be written as:

$$Q_{mass}(x,t) = \frac{2M_{clamp}}{L} \cdot \omega_0^2 \cdot \sum_{l=3}^{\infty} \left[ g_0 \cdot g_n \cdot n^4 \cdot C_n \cdot P_n(x) \right] \cdot e^{i(\alpha_l \cdot \varphi_n^*)}$$

This formulation is similar to the mass inertia force of the beam:

$$\rho \cdot b \cdot h \cdot \omega_0^2 \cdot \sum_{l=3}^{\infty} \left[ C_n \cdot P_n(x) \right] \cdot e^{i(\alpha_l \cdot \varphi_n^*)}$$

Therefore the resulting equations can be stated as:

$$C_n \cdot \left[ \frac{2}{M_{beam}} \cdot \frac{F_0}{g_0} \cdot \cos(\varphi - \varphi_n^*) - \omega_0^2 \cdot \gamma_n \cdot \cos(\varphi_n^*) \right] = 0$$

$$C_n \cdot \left[ \frac{2}{M_{beam}} \cdot \frac{F_0}{g_0} \cdot \sin(\varphi - \varphi_n^*) - \omega_0^2 \cdot \gamma_n \cdot \sin(\varphi_n^*) \right] = 0$$

---

3 In Part II of this series a solution is given which can be used in Excel files.
\[ \gamma_n = 1 + \frac{2M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A] \cdot g_0 \cdot g_n \cdot n^4 \] (205)

The unknown parameters \( g_0 \) and \( g_n \) can be determined as follows:

The first step is to calculate the deflection at the inner clamp \( V[A] \) with the starting values \( g_0 = 1 \) and \( g_n = 1 \).

The ratio of the calculated values for the total deflection \( V \) and the first term \( V(A;n=1) \) of the series will give a better estimate \( g_0^* \) of the parameter \( g_0 \).

\[ g_0^* = \frac{\hat{V}(A)}{C_1 \cdot P_1(A)} = \frac{\hat{V}(A)}{V(A, n = 1)} \] (206)

The next step is to calculate the coefficient \( C_n \) for each term of the series:

\[ C_n = \frac{\hat{V}(A, n = n)}{P_n(A)} \] (207)

The third step is the calculation of a better approximation for the parameter \( g_n \) (of course \( g_1 \) will be equal to 1):

\[ g_n^* = \frac{\hat{V}(A)}{g_0^* \cdot n^4 \cdot C_n \cdot P_1(A)} \] (208)

Now the calculations are performed again with the new values for \( g_0^* \) and \( g_n^* \) until these values and the obtained deflection \( V[A] \) do not change significantly. The only restriction is that the phase lag remains 'constant' along the beam, which is not true in principle.

With the final obtained values for the parameters \( g_0^* \) and \( g_n^* \), the deflection for an arbitrarily chosen point \( X \) can be calculated.
5.3 Back calculation procedure

For the back calculation procedure we will only use the first order approximation, being the first term of the series. In that case \( V[A] \) is approximated by \( C_1 \times P_1(A) \) \( (g_0=g_1=1) \) and \( \phi_1^* \) is taken equal to \( \phi^* \).

The resulting equations are:

\[
C_1 \cdot \left[ \xi^2_i \cdot \cos (\phi - \phi^*_i) - \omega_0^2 \cdot \left[ 1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A] \right] \cdot \cos (\phi^*_i) \right] = \frac{2 \cdot F_0}{M_{\text{beam}}} \quad (209)
\]

\[
C_1 \cdot \left[ \xi^2_i \cdot \sin (\phi - \phi^*_i) + \omega_0^2 \cdot \left[ 1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A] \right] \cdot \sin (\phi^*_i) \right] = 0 \quad (210)
\]

These relations are identical of form compared to the viscous-elastic case without the extra mass \( M_{\text{clamp}} \), if \( \omega_0 \) is replaced by:

\[
\omega_0 \Rightarrow \omega_0 \cdot \sqrt{1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A]} \quad (211)
\]

The phase lag \( \phi_1^* \) will be:

\[
\tan (\phi^*_i) = \frac{\sin (\phi)}{\cos (\phi) \cdot \frac{\omega_0^2}{\xi^2_i} \cdot \left[ 1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A] \right]} \quad (212)
\]

Adopting \( \gamma = 1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot P_1[A] \) the coefficient \( C_1 \) is given by:

\[
C_1 = \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot \hat{S}_{\text{mix}} \cdot I} \cdot \left[ \frac{1}{\sqrt{1 - 2 \cdot \frac{\omega_0^2}{\xi^2_i} \cdot \gamma \cdot \cos (\phi) + \left( \frac{\omega_0^2}{\xi^2_i} \cdot \gamma \right)^2}} \right] \quad (213)
\]

The deflection along the beam is approximated straight forward by:

\[
V(x,t) = \frac{2 \cdot F_0 \cdot L^3}{\pi^4 \cdot \hat{S}_{\text{mix}} \cdot I} \cdot \frac{\sin (\pi \cdot \frac{A}{L}) \cdot \sin (\pi \cdot \frac{x}{L}) \cdot e^{i(\omega_0 t - \phi^*_i)}}{\sqrt{1 - 2 \cdot \cos (\phi) \cdot \frac{\omega_0^2}{\xi^2_i} \cdot \gamma + \left( \frac{\omega_0^2}{\xi^2_i} \cdot \gamma \right)^2}} \quad (214)
\]

Again this approximation is equal to the solution for a viscous-elastic single mass-spring system.

In the following text we will use for short writing the following abbreviations (see also paragraph 4.4):

\[
R(x) = \frac{\pi^4}{2 \cdot P_1(x)} = \frac{\pi^4}{2 \cdot \sin (\pi \cdot \frac{A}{L}) \cdot \sin (\pi \cdot \frac{x}{L})} \quad (215)
\]
\[ R'(x) = \frac{12 \cdot L}{A} \cdot \frac{1}{3 \cdot \frac{x}{L} - 3 \cdot \frac{x^2}{L^2} \cdot \frac{A^2}{L^2}} \] (216)

The approximation (equation 214 with \( \phi_1 = \phi^* \)) can be rewritten as:

\[
V(x, t) = \frac{F_0 \cdot L^3}{R(x) \cdot S_{mix} \cdot I} \cdot \frac{e^{i(\omega_0 t - \phi^*)}}{\sqrt{1 - 2 \cdot \cos(\phi) \cdot \frac{\omega_0^4}{\xi_1^2} \cdot \gamma + \frac{\omega_0^4}{\xi_1^4} \cdot \gamma^2}}
\] (217)

in which the coefficients \( \gamma \) and \( \xi \) are given by:

\[
\gamma = 1 + \frac{\pi^4}{R(A)} \cdot \frac{M_{clamp}}{M_{beam}}; \quad \xi_1^2 = \frac{\pi^4 \cdot S_{mix} \cdot I}{L^3} \cdot \frac{1}{M_{beam}}
\] (218)

The equivalent mass-spring system is therefore characterized by:

\[
K = \frac{S_{mix} \cdot I}{L^3} \cdot R(x); \quad M_{equiv.} = \frac{K \cdot \gamma}{\xi_1^2} = \frac{R(x)}{\pi^4} \cdot M_{beam} + \frac{R(A)}{R(A)} \cdot M_{clamp}
\] (219)

If the displacement is measured at \( x = L/2 \) this becomes:

\[
M_{equiv.} = \frac{I}{2 \cdot \sin(\pi \cdot \frac{A}{L})} \cdot M_{beam} + \sin(\pi \cdot \frac{A}{L}) \cdot M_{clamp}
\] (220)

In case of the DWW specifications (\( A = 0.135 \) m and \( L = 0.400 \) m) this leads to:

\[
M_{equiv.} = 0.5731 M_{beam} + 0.8752 M_{clamp}
\]

if the deflection is measured in the centre of the beam (\( x = L/2; R(x)=55.8221 \)).

As shown in chapter 4 the modified first order approximation (using \( R' \) instead of \( R \)) will lead to better results and is preferable for the back calculation procedure.

If we replace in the equations above the term R(x) by \( R'(x) \), we will obtain in case of the DWW specifications the following values for the deflection measurement in the centre (\( R'(L/2)=55.897; R'(A)=63.848 \)):

\[
M_{equiv.} = 0.5738 M_{beam} + 0.8755 M_{clamp}
\]

Although not quite correct these formulations have been used in the advised back calculations procedures (chapter 8) even when the beam length \( L_{tot} \) is longer than the distance \( L \) between the two outer clamps.
6. Dynamic Bending of an Elastic Beam with a length $L_{\text{tot}}$ longer than the length $L$ between the two outer clamps.

The coordinate system is chosen as follows:

$x = 0$ : beginning of the beam ; $x = L_{\text{tot}}$ : end of the beam

outer clamps $x = \Delta = \frac{L_{\text{tot}} - L}{2}$ ; $x = L_{\text{tot}} - \Delta$

inner clamps $x = \Delta + A$ ; $x = \Delta + B = L_{\text{tot}} - \Delta - A$

$L$ = length between outer clamps ; $L_{\text{tot}}$ = actual beam length

The differential equation is given by:

$$E \cdot I \cdot \frac{d^4}{dx^4} V (x, t) + \rho \cdot b \cdot h \cdot \frac{d^2}{dt^2} V (x, t) = Q(x, t)$$

In this case the two point loads at the inner clamps have to be developed in a force distribution along the whole beam of length $L_{\text{tot}}$ (equation 223) as well as the reaction forces at the two outer clamps (equation 224); $Q(x,t)=Q_1(x,t)+Q_2(x,t)$:

$$\hat{Q}_1(x) = \frac{2 \cdot F_0}{L_{\text{tot}}} \sum_{n=1,3,5} \sin \left( n \cdot \frac{\pi}{L_{\text{tot}}} \cdot A + \frac{\Delta}{L_{\text{tot}}} \right) \sin \left( n \cdot \frac{\pi}{L_{\text{tot}}} \cdot \frac{x}{L_{\text{tot}}} \right)$$

$$\hat{Q}_2(x) = -\frac{2 \cdot F_0}{L_{\text{tot}}} \sum_{n=1,3,5} \sin \left( n \cdot \frac{\pi}{L_{\text{tot}}} \cdot \frac{\Delta}{L_{\text{tot}}} \right) \sin \left( n \cdot \frac{\pi}{L_{\text{tot}}} \cdot \frac{x}{L_{\text{tot}}} \right)$$

The homogenous differential equation is given by:

$$E \cdot I \cdot \frac{d^4}{dx^4} V (x, t) + \rho \cdot b \cdot h \cdot \frac{d^2}{dt^2} V (x, t) = 0$$

We are looking for solutions of the form:

$$V(x, t) = V(x) \cdot e^{i \omega t} = \sum \left[ V_a(x) + V_c(x) + V_d(x) \right] \cdot e^{i \omega t}$$

Invoking this type of solution leads to:

$$E \cdot I \cdot \frac{d^4}{dx^4} V_a(x) - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot V_a(x) = \hat{Q}(x)$$

$$E \cdot I \cdot \frac{d^4}{dx^4} V_{c,d}(x) - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot V_{c,d}(x) = 0$$

Introducing the parameters $\xi_0$, $\xi_n$, $\beta_0$ and $\beta_n$:

$$\xi_0^2 = \frac{E \cdot I}{\rho \cdot b \cdot h}$$

$$\xi_n^2 = \frac{n^4 \cdot \pi^4}{L_{\text{tot}}^4} \cdot \xi_0^2 = \frac{L_{\text{tot}}^4 \cdot \omega_n^2}{\omega_0^2}$$

$$\beta_0 = \frac{\omega_0}{\sqrt{\xi_0}}$$

$$\beta_n = \sqrt{\frac{\xi_n}{\xi_0}} \cdot \omega_0 \cdot \sqrt{L_{\text{tot}} \cdot \omega_n}$$

we obtain for the non-homogenous differential equation:
\[ \frac{\xi_n^2 \cdot L_{tot}^4}{n^4 \cdot \pi^4} \cdot V_{a,x,x,x} - \omega_0^2 \cdot V_a = \frac{1}{\rho \cdot b \cdot h} \cdot \hat{Q}(x) \]  

and for the homogenous differential equation:

\[ \frac{\xi_n^2}{\omega_0^2} \cdot V_{c,d,x,x,x} - \omega_0^2 \cdot V_{c,d} = 0 \]

We are only interested in solutions which are symmetrical around the center \((x=L_{tot}/2)\). For \(Q(x,t)=Q(x) \times e^{i\omega t}\) we obtain with \(n\) is a odd number:

\[ V_a(x) = A_n \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L_{tot}} \right) \text{with } n = \text{odd} \]

\[ A_n = \frac{2 \cdot F_0 \cdot L_{tot}^3}{\pi^4 \cdot E \cdot L} \cdot \sin \left( n \cdot \pi \cdot \frac{A + \Delta}{L_{tot}} \right) - \sin \left( n \cdot \pi \cdot \frac{\Delta}{L_{tot}} \right) \]

\[ n^4 \cdot \left[ 1 - \frac{\omega_0^2}{\xi_n^2} \right] \]

For \(Q(x,t)=0\) we have two independent solutions given by:

\[ V_c(x) = C_n \cdot \cos \left( \beta_0 \cdot \left[ x - L_{tot} \right] \right) \]

\[ V_d = D_n \cdot \cosh \left( \beta_0 \cdot \left[ x - L_{tot} \right] \right) \]

The first solution (infinite series) with the coefficient \(A_n\) satisfies the differential equation with the force distribution \(Q(x,t)\). The second and third solutions satisfy the homogenous differential equation. The boundary conditions are:

\[ \text{for } x = 0 \text{ and for } x = L_{tot}: \quad \frac{d^2 V(x)}{dx^2} = 0 \]

\[ \text{for } x = \Delta \text{ and for } x = L_{tot} - \Delta: \quad V(x) = 0 \]

Because the chosen functions are symmetrical round the center, it yields that if these boundary conditions are fulfilled for \(x=x_1\), the boundary conditions at \(x=L_1-x_1\) are also satisfied. For the solution of the problem we need only one particular solution of the differential equation and two general solutions of the homogenous differential equation. From the boundary condition at \(x=0\) (and \(x=L_{tot}\)) it follows that \(C_n\) and \(D_n\) are related according to:

\[ D_n = C_n \cdot \frac{\cos \left( \frac{L_{tot} \cdot \beta_0}{2} \right)}{\cosh \left( \frac{L_{tot} \cdot \beta_0}{2} \right)} = \lambda_0 \cdot C_n \]
From the boundary condition at $x=\Delta$ (and $x=L_{\text{tot}}-\Delta$) it follows that $C_n$, $D_n$ and $A_n$ are related according to:

$$A_n \cdot \sin \left( n \cdot \frac{\pi \cdot \Delta}{L_{\text{tot}}} \right) + C_n \cdot \cos \left( \beta_0 \cdot \left[ \Delta - \frac{L_{\text{tot}}}{2} \right] \right) + D_n \cdot \cosh \left( \beta_0 \cdot \left[ \Delta - \frac{L_{\text{tot}}}{2} \right] \right) = 0$$

Inserting equation 236 into equation 237 will give $C_n$ as a function of $A_n$. The value for $A_n$ is given by equation 232.
7. Dynamic Bending of an Visous- Elastic Beam with a length $L_{tot}$ longer than the length $L$ between the two outer clamps.

The coordinate system is chosen again as:

$x = 0 : \text{beginning of the beam} ; x = L_{tot} : \text{end of the beam}$

outer clamps $x = \Delta = \frac{L_{tot} - L}{2} ; x = L_{tot} - \Delta$

inner clamps $x = \Delta + A ; x = \Delta + B = L_{tot} - \Delta - A$

$L = \text{length between outer clamps} ; L_{tot} = \text{actual beam length}$

The differential equation is now given by:

$$S_{mix} \cdot e^{i\phi} \cdot I \cdot \frac{d^4}{dx^4} V(x, t) + \rho \cdot b \cdot h \cdot \frac{d^2}{dt^2} V(x, t) = Q(x, t)$$ \hspace{1cm} (238)

The point loads at the inner and outer clamps are developed in a force distribution along the whole beam of length $L_{tot}$:

$$\hat{Q}_1(x) = \frac{2 \cdot F_0}{L_{tot}} \cdot \sum_{n=1,3,5}^\infty \sin\left(n \cdot \pi \cdot \frac{A + \Delta}{L_{tot}}\right) \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L_{tot}}\right)$$ \hspace{1cm} (240)

$$\hat{Q}_2(x) = -\frac{2 \cdot F_0}{L_{tot}} \cdot \sum_{n=1,3,5}^\infty \sin\left(n \cdot \pi \cdot \frac{\Delta}{L_{tot}}\right) \cdot \sin\left(n \cdot \pi \cdot \frac{x}{L_{tot}}\right)$$ \hspace{1cm} (241)

The homogenous differential equation is given by equation 242 and the required solution form by equation 243:

$$S_{mix} \cdot e^{i\phi} \cdot I \cdot \frac{d^4}{dx^4} V(x, t) + \rho \cdot b \cdot h \cdot \frac{d^2}{dt^2} V(x, t) = 0$$ \hspace{1cm} (242)

$$V(x, t) = V_a(x) \cdot e^{i\omega_0 t} + V_c(x) + V_d(x) \cdot e^{i\omega_n t}$$ \hspace{1cm} (243)

The deflection $V_a(x)$ satisfies the load conditions. $V_c(x)$ and $V_d(x)$ are two solutions of the homogeneous differential equation, needed to satisfy the boundary conditions.

Invoking this type of solution leads to:

$$S_{mix} \cdot e^{i\phi} \cdot I \cdot \frac{d^4}{dx^4} V_a(x) - \rho \cdot b \cdot h \cdot \omega_0^2 \cdot V_a(x) = \hat{Q}(x)$$ \hspace{1cm} (244)

$$S_{mix} \cdot e^{i\phi} \cdot I \cdot \frac{d^4}{dx^4} V_{c,d}(x) - \rho \cdot b \cdot h \cdot \omega_n^2 \cdot V_{c,d}(x) = 0$$ \hspace{1cm} (245)

Introducing the parameters $\xi_0, \xi_n, \beta_0$ and $\beta_n$ as done in chapter 6 ($E \to S_{mix}$):

$$\xi_0^2 = \frac{S_{mix} \cdot I}{\rho \cdot b \cdot h} ; \quad \xi_n^2 = \frac{n^4 \cdot \pi^4}{I_{tot}^4} \cdot \omega_0^2 ; \quad \beta_0 = \sqrt{\frac{\omega_0}{\xi_0}} \quad \text{and} \quad \beta_n = \sqrt{\frac{\omega_n}{\xi_n}} \cdot \frac{L_{tot}}{L} \cdot \frac{\omega_0}{\omega_n}$$ \hspace{1cm} (246)

we obtain for the non-homogenous differential equation:
\[ e^{i\varphi} \frac{x^2}{n^4 \cdot \pi^2} \cdot V_{a,x,...} - \omega_0^2 \cdot V_a = \frac{1}{\mu \cdot b \cdot h} \cdot \dot{Q}(x) \]

and for the homogeneous differential equation:

\[ e^{i\varphi} \frac{x^2}{\xi_0^2} \cdot V_{c,x,...} - \omega_0^2 \cdot V_{c,d} = 0 \]

We are only interested in solutions which are symmetrical around the center \((x = L_{tot}/2)\).

For \(Q(x,t) = Q(x) \times e^{i\omega_0 t}\) we obtain with \(n\) is an odd number (see chapter 4):

\[
V_a(x) = A_n \cdot \sin \left( n \cdot \pi \cdot \frac{x}{L_{tot}} \right) \cdot e^{\frac{1}{2} \arctan \left( \frac{\xi_\varphi}{\xi_\epsilon} \cdot \sin(\varphi) \right)} \quad \text{with } n \text{ is odd : } 1, 3, 5,...
\]

\[
A_n = \frac{2 \cdot F_0 \cdot L_{tot}^3}{\pi^3 \cdot S_{mix} \cdot I} \cdot \frac{\sin \left( n \cdot \pi \cdot \frac{A + \Delta}{L_{tot}} \right)}{n^4 \cdot \sqrt{1 - 2 \cdot \cos(\varphi) \cdot \frac{\omega_0^2}{\xi_n^2} + \frac{\omega_0^4}{\xi_n^4}}}
\]

The general solutions of the homogeneous differential equation are given by (see chapter 4.1):

\[
V_{c,d}(x) = e^{\pm \beta_n e^{i\alpha} \cdot x} ; e^{-i \beta_n e^{i\alpha} \cdot x} ; e^{\pm i \beta_n e^{i\alpha} \cdot x} \quad \alpha = -\frac{\varphi}{4}
\]

The two required symmetrical solutions are:

\[
V_c(x) = \frac{C_n}{2} \left[ e^{i \beta_n e^{i\alpha} \cdot (x \cdot \frac{L_{tot}}{2})} + e^{-i \beta_n e^{i\alpha} \cdot (x \cdot \frac{L_{tot}}{2})} \right]
\]

\[
V_d(x) = \frac{D_n}{2} \left[ e^{i \beta_n e^{i\alpha} \cdot (x \cdot \frac{L_{tot}}{2})} + e^{-i \beta_n e^{i\alpha} \cdot (x \cdot \frac{L_{tot}}{2})} \right]
\]

These equations can be rewritten as:

\[
V_c(x) = C_n \cdot \left\{ \begin{array}{l}
+\cos(\beta_n \cos[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \cdot \cosh(\beta_n \sin[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \\
-i \cdot \sin(\beta_n \cos[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \cdot \sinh(\beta_n \sin[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right))
\end{array} \right\}
\]

\[
V_d(x) = D_n \cdot \left\{ \begin{array}{l}
+\cos(\beta_n \sin[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \cdot \cosh(\beta_n \cos[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \\
+i \cdot \sin(\beta_n \sin[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right)) \cdot \sinh(\beta_n \cos[\alpha] \cdot \left( x - \frac{L_{tot}}{2} \right))
\end{array} \right\}
\]

If the phase lag \(\varphi\) goes to zero (elastic case), the solutions are equal to those used in chapter 6.

The first solution (infinite series) with the coefficient \(A_n\) satisfies the differential equation with the force distribution \(Q(x,t)\). The second and third solutions satisfy the homogeneous differential equation. The boundary conditions are:
Because the chosen functions are symmetrical around the center, it means that if these boundary conditions are fulfilled for $x=x_1$, the boundary conditions at $x=L_{tot}-x_1$ are also satisfied. For the solution of the problem we need only one particular solution of the differential equation and two general solutions of the homogenous differential equation. From the boundary condition at $x=0$ (and $x=L_{tot}$) it follows that $C_n$ and $D_n$ are related according to:

$$D_n = C_n \cdot \frac{\cos \left( \frac{L_{tot} \cdot \beta_{\theta} \cdot e^{i\alpha}}{2} \right)}{\cosh \left( \frac{L_{tot} \cdot \beta_{\alpha} \cdot e^{i\alpha}}{2} \right)} = \lambda_0 \cdot C_n$$

The coefficient $\lambda_0$ is of course now complex.

From the boundary condition at $x=\Delta$ (and $x=L_{tot}-\Delta$) it follows that $C_n$, $D_n$ and $A_n$ are related according to:

$$A_n \sin \left( \frac{n \cdot \pi \cdot \Delta}{L_{tot}} \right) \cdot e^{i \varphi_n} + C_n \cos \left( \beta_{\theta} \left[ \Delta - \frac{L_{tot}}{2} \right] \cdot e^{i\alpha} \right) + D_n \cosh \left( \beta_{\alpha} \left[ \Delta - \frac{L_{tot}}{2} \right] \cdot e^{i\alpha} \right) = 0$$

$$\text{with} \quad \tan \left( \varphi_n^* \right) = \frac{\sin \left( \varphi \right)}{\cos \left( \varphi \right) - \frac{\omega_n}{\omega_n^*}}$$

These equations have been used to compute the centre deflection for the DWW equipment ($L_{tot} = 0.450 \text{ m} ; L = 0.400 \text{ m} \text{ and } A = 0.135 \text{ m}$) as a function of the frequency $f_0$. The results are compared with the centre deflections and phase lags for a beam of which the actual length equals the distance $L=0.400 \text{ m}$ between the two outer clamps. The deflections are given in figures 7.1 and the phase lags $\varphi_n^*$ in figure 7.2 for a complex stiffness modulus ($S_{\text{mix}} = 2000 \text{ MPa}; \varphi = 20^\circ$).
Figure 7.1  Centre deflections as a function of the frequency for two beam lengths ($L_{tot}=450$ mm and $L_{tot}=400$ mm). $S_{mix}=2000$ MPa; $\phi=20^\circ$.

Figure 7.2  Calculated phase lags in the centre as a function of the frequency for two beam lengths ($L_{tot}=450$ mm and $L_{tot}=400$ mm); $S_{mix}=2000$ MPa; $\phi=20^\circ$. 
Back calculation

Although the complete solution does not resemblance anymore the solution for a viscous-elastic mass-spring system, it is still worthwhile to compare it with such a system. In the equivalent mass-spring system we take the distance \( L \) between the two outer clamps for the calculation of \( K \) instead of the actual length \( L_{\text{tot}} \), so ignoring the overhanging distance \( \Delta \). However, For the calculation of the equivalent mass \( M_{\text{equiv}} \), we do use the total weight of the actual beam \( M_{\text{beam}} \).

Using this notation the formulas for the back calculation are identical to the ones used in chapter 4 (equations 177 and 178). The equivalent mass and spring may be calculated according to equation 173 (‘straight forward’ 1st order approximation) or equation 174 (modified 1st order approximation). The differences in the back calculated values for the stiffness modulus \( S_{\text{mix}} \) and phase lag \( \varphi \) are negligible between the two types of calculations. The results for a beam with a stiffness modulus \( S_{\text{mix}} = 2000 \text{ MPa} \) and a phase lag \( \varphi = 20^\circ \) are given in figures 7.3 and 7.4 (based on a calculated centre deflection).

![Back calculated stiffness modulus as a function of the frequency using a 1st order approximation.](image)

Figure 7.3

![Back calculated phase lag as a function of the frequency using a 1st order approximation.](image)

Figure 7.4
Strain calculations

However, if the occurring strain in the edge of the beam has to be calculated, the modified 1st order approximation is superior (chapter 4; equation 176). In figure 7.5 the results are given for a beam with a stiffness modulus $S_{\text{mix}}$ of 2000 MPa and a phase lag $\phi$ of 20° (see for the other parameters the legend of figure 7.1). The exact calculated value (equation 175) and the value using the (modified) coefficient of equation 176 are the same, while the normal 1st order approximation is slightly higher.

Figure 7.5 Comparison of strains
**Influence of extra moving masses**

The exact solution is hard to obtain\(^4\), but in the same way as described in chapter 5 a kind of an approximation can be made.

The development of the extra moving force is now given by:

\[
Q_{\text{mass}} = \frac{2 \cdot M_{\text{clamp}} \cdot \omega^2}{L_{\text{tot}}} \sum \left[ \hat{\mathcal{V}} \left( A + \Delta \right) \cdot e^{i \left( \alpha_n \cdot \varphi \left(x + \Delta \right) \right)} \cdot T_n \left( x \right) \right]
\]

in which \(T_n(x)\) is given by:

\[
T_n \left( x \right) = \sin \left( n \cdot \pi \cdot \frac{x}{L_{\text{tot}}} \right) \cdot \left[ \sin (n \cdot \pi \cdot \frac{A + \Delta}{L_{\text{tot}}}) - \sin (n \cdot \pi \cdot \frac{\Delta}{L_{\text{tot}}} ) \right]
\]

The deflection \(V(x,t)\) will be of the form:

\[
V \left( x , t \right) = \left[ V_a \left( x \right) + V_c \left( x \right) + V_d \left( x \right) \right] \cdot e^{i \varphi t}
\]

in which \(V_a(x)\) will be given by (see equation 249):

\[
V_a \left( x \right) = A_n \cdot T_n \left( x \right) \cdot e^{i \varphi_{a}(x)}
\]

The first assumption will be that the deflection can be written as:

\[
V \left( x , t \right) = \sum s_n \cdot C_n \cdot T_n \left( x \right) \cdot e^{i \left( \alpha_n \cdot \varphi_{n}(x) \right)}
\]

An approximation for the amplitude at the clamp \(x=A+\Delta\) will be:

\[
\hat{V}(A+\Delta) = g_{o} \cdot s_{1} \cdot C_{1} \cdot T_{1}(A+\Delta) = g_{o} \cdot f_{n} \cdot s_{n} \cdot C_{n} \cdot T_{1}(A+\Delta); \text{ with } f_{n} = \frac{s_{1} \cdot C_{1}}{s_{n} \cdot C_{n}} = g_{n} \cdot n^{4}
\]

So, for the 'mass factor' we will find (see equation 205):

\[
\gamma_{n} = 1 + \frac{2 \cdot M_{\text{clamp}}}{M_{\text{beam}}} \cdot T_{1} \left[ A + \Delta \right] \cdot g_{o} \cdot g_{n} \cdot n^{4}
\]

For the determination of the coefficients \(g_{o}\) and \(g_{n}\) we follow the procedure as outlined in paragraph 5.2 (equations 206 to 208) using the function \(T_n(x)\) instead of \(P_n(x)\).

*This procedure is used for the forward calculation of the deflections if the complex modulus is given. For the back calculation of the stiffness modulus from measured deflections, the procedure as described in chapter 8 is advised.*

---

\(^4\) In Part II of this series the exact solution is given
An Overview of the Advised Equations for the Back calculation Procedure

8.1 General Formulas

The advised back calculation procedure is based on the modified first order approximation as discussed in chapter 4.4. In this approximation the deflection at a point x is taken equal to the deflection of an equivalent single mass-spring system. In the formulas below the parameters L and A represent the distance between the two outer clamps and the distance between the outer and inner clamp respectively. This mass-spring system is characterized by the following parameters:

\[
K'(x) = \frac{S_{\text{mix}} \cdot I}{L^3} \cdot R'(x) \quad 265
\]

\[
M'_{\text{equiv}} = \frac{R'(x)}{\pi^4} \cdot M_{\text{beam}} + \frac{R'(x)}{R'(A)} \cdot M_{\text{clamp}} \quad 266
\]

\[
R'(x) = \frac{12 \cdot L}{A} \cdot \frac{1}{3 \cdot \frac{x}{L} - 3 \cdot \frac{x^2}{L^2} - \frac{A^2}{L^2}} \quad 267
\]

The stiffness modulus \(S_{\text{mix}}\) is back calculated from the deflection measurement by:

\[
S_{\text{mix}} = \frac{F_0 \cdot L^3}{\dot{V} \cdot [x] \cdot R'(I)} \cdot \sqrt{1 + 2 \cdot \cos(\varphi^* [x]) \cdot Z[x] + \left(Z[x]\right)^2} \quad 268
\]

The phase lag \(\varphi\) is back calculated by:

\[
\tan(\varphi_{\text{backc}}) = \frac{\sin(\varphi^* [x])}{\cos(\varphi^* [x]) + Z[x]} \quad 269
\]

in which \(Z[x]\) is given by:

\[
Z[x] = M'_{\text{equiv}} \cdot \frac{\dot{V} [x]}{F_0 \cdot \varphi_0^3} \quad 270
\]

If in the calculations the arctangent function (atan) is used for the determination of \(\varphi\) and \(\varphi^*\) one can obtain negative values. This is due to the fact that the arctangent function is defined from \(-\pi/2\) to \(+\pi/2\). Because both \(\varphi\) and \(\varphi^*\) have to be positive (see chapter 4), the correct value is given by \(\varphi = \pi + \varphi\) if \(\varphi\) is negative.
**Strain Calculations**

The occurring strain at the coordinate \( x \) in the edge of the beam is given by:

\[
\varepsilon [ x ] = \frac{h \cdot A}{4 \cdot L^3} \cdot R' [ x ] \cdot V [ x ]
\]

Because the maximum strain will occur in the centre \((x=L/2)\) it is advised to measure the deflection in the centre. However, the difference will be small if the deflection is measured at the inner clamp \((x=A)\).

**Back calculation**

For the back calculation the actual length \( L_{tot} \) of the beam is not relevant, except of course for the calculation of the weight of the beam: \( M_{beam} = L_{tot} \cdot b \cdot h \cdot \rho \).

Although no large errors are introduced, it is not advised to correct measured deflections at the inner clamps to fictive measured deflections at the centre in order to use a back calculation procedure which is based on centre deflections.

If the extra moving masses at the inner clamps are of the same order as the weight of the beam, it is advised to check the back calculated values. The easiest way is to use the procedure as outlined in chapter 5, using the total weight of the beam but taking \( \Delta = 0 \) \((L_{tot} = L)\). In this way an indication is obtained of the introduced error by the first order approximation. Of course the procedure outlined in chapter 7 \((L_{tot} > L)\) can also be used but this procedure is more complex.

8.2 **ASTM Specifications**  \( A = L/3 \)  \((400/3 \text{ mm})\)

**Measurements at the inner clamps**

The parameter \( R' \) is given by: \( R'(x) = R'(A) = 64,8 \)

The equivalent mass is given by: \( M_{equiv.} = 0,6652 \ M_{beam} + 1,0 \ M_{clamp} \)

**Measurements at the centre of the beam**

The parameter \( R' \) is given by: \( R'(x) = R'(L/2) = 56,3478 \)

The equivalent mass is given by: \( M_{equiv.} = 0,5785 \ M_{beam} + 0,8696 \ M_{clamp} \)
8.3 DWW Specifications  \( A=135 \text{ mm}; \, L=400 \text{ mm} \)

*Measurements at the inner clamps*

The parameter \( R' \) is given by: \( R'(x) = R'(A) = 63,8484 \)

The equivalent mass is given by: \( M_{\text{equiv.}} = 0,6555 \, M_{\text{beam}} + 1,0 \, M_{\text{clamp}} \)

*Measurements at the centre of the beam*

The parameter \( R' \) is given by: \( R'(x) = R'(L/2) = 55,8967 \)

The equivalent mass is given by: \( M_{\text{equiv.}} = 0,5738 \, M_{\text{beam}} + 0,8755 \, M_{\text{clamp}} \)