Theory of the Four Point Dynamic Bending Test

Part IV: Pure Bending & Shear Deformation

IMPORTANT NOTICE March 8 & 9; 2007

In the original report the coefficient in the formula for the effective cross area (denoted by \( R\{B, H\} \), for the shear force), is taken equal to 2/3. Just before the 1st European 4PB workshop on March 8 & 9; 2007 it was concluded, based on ABACUS FEM calculations, that a value of 0.85 was more appropriated and leads to: \( R\{B, H\} = 0.85.B.H \)
THEORY OF THE
FOUR POINT DYNAMIC
BENDING TEST

PART IV: PURE BENDING & SHEAR DEFORMATION

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Abstract

This report deals with the commonly neglected influences of shear forces on the measured deflections in the dynamic four point bending test. The report forms the fourth part in the series of reports on the “Theory of the Four Point Dynamic Bending Test”. It is world wide adopted that for a bending test in which the ratio of the (effective) length or span of the beam and the height of the beam is above a factor 8 the deflection due to shear forces can be neglected. This judgement is based on a comparison between the differential equations for pseudo-static bending tests. In this report the complete analytical solutions are derived for cyclic bending test conditions. It is shown that the deflection part due to shear forces is around 5% of the total deflection. For a good understanding of the theory it is advised to use Part I and II of the series on the Theory of the Four Point Bending Test next to this report.

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DISCLAIMER

This working paper is issued to give those interested an opportunity to acquaint themselves with progress in this particular field of research. It must be stressed that the opinions expressed in this working paper do not necessarily reflect the official point of view or the policy of the director-general of the Rijkswaterstaat. The information given in this working paper should therefore be treated with caution in case the conclusions are revised in the course of further research or in some other way. The Kingdom of the Netherlands takes no responsibility for any losses incurred as a result of using the information contained in this working paper.
1. GENERAL BENDING THEORY OF A RECTANGULAR BEAM

The bending theory for a rectangular beam is given by two differential equations:

\[
\rho B H \frac{\partial^2}{\partial x^2} \left[ V_b \{x,t\} + V_s \{x,t\} \right] - \frac{\partial}{\partial x} D = Q \{x,t\} \quad [1_{a,b}]
\]

\[- \rho I \frac{\partial}{\partial x} \frac{\partial^3}{\partial x^3} V_b \{x,t\} - \frac{\partial}{\partial x} M + D = 0 \]

In which:
- \( V_b \) = Deflection due to pure bending
- \( V_s \) = Deflection due to shear forces
- \( B \) = Width of beam
- \( H \) = Height of beam
- \( L_{tot} \) = Total length of beam
- \( L \) = Effective length = distance between the two outer supports
- \( A \) = Distance between outer and inner supports
- \( \Delta \) = Distance from \( x=0 \) to first outer support (=\((L_1-L)/2\))
- \( F_o \) = Applied force at the two inner supports (clamps)
- \( I \) = Moment of the beam \((B H^3/12)\)
- \( D \) = Shear force
- \( M \) = Bending moment
- \( Q \) = Force distribution along the beam
- \( E \) = Stiffness modulus of beam
- \( G \) = Shear modulus \((E/(2(1+\mu)))\)
- \( \rho \) = Density
- \( \mu \) = Poisson ratio

The moment \( M \) is related to the deflection \( V_b \) by:

\[
M = - E I \frac{\partial^2}{\partial x^2} V_b \{x,t\} \quad [2]
\]

And the shear force \( D \) is related to the shear deflection \( V_s \) by:

\[
D = + G \Re \{B,H\} \frac{\partial}{\partial x} V_s \{x,t\} \quad \text{with} \quad \Re \{B,H\} = \text{Effective Cross Area} \quad [3]
\]

Integration of equation \([1b]\) over \( x \) leads to a relationship between the shear deflection \( V_s \) and the bending deflection \( V_b \):

\[
V_s \{x,t\} = \frac{I}{G \Re \{B,H\}} \left[ \rho \frac{\partial^2}{\partial x^2} V_b \{x,t\} - E \frac{\partial}{\partial x} \frac{\partial^3}{\partial x^3} V_b \{x,t\} + \Psi \{t\} \right] \quad [4]
\]

in which \( \Psi \{t\} \) is a yet unknown function in the time \( t \) only.
Differentiation of equation [1b] and adding to equation [1a] leads to a relationship between the total deflection \( V_t \) and the bending deflection \( V_b \):

\[
\rho B H \frac{\partial^2}{\partial t^2} V_t(x,t) - \rho I \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_b(x,t) + E I \frac{\partial^4}{\partial x^4} V_b(x,t) = Q(x,t) \tag{5}
\]

Using equation [4] in equation [5] leads to the desired relationship between the bending deflection \( V_b \) and the force distribution \( Q \):

\[
\rho B H \frac{\partial^2}{\partial t^2} V_b(x,t) - \rho I \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_b(x,t) + E I \frac{\partial^4}{\partial x^4} V_b(x,t) + \frac{\rho B H I}{G \mathcal{R} \{B, H\}} \left( \rho \frac{\partial^4}{\partial t^4} V_b(x,t) - E \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_b(x,t) + \frac{d^2}{dt^2} \Psi(t) \right) = Q(x,t) \tag{6}
\]

Elimination of \( V_b \) by substituting equation 1a into equation 1b after two derivations with respect to the time \( t \) leads to:

\[
\frac{E I}{\rho B H} \left[ \frac{d^2}{dx^2} Q(x,t) - \rho B H \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_s(x,t) + G \mathcal{R} \{B, H\} \frac{\partial^4}{\partial x^4} V_s(x,t) \right] - \frac{\rho I}{\rho B H} \left[ \frac{d^2}{dt^2} Q(x,t) - \rho B H \frac{\partial^4}{\partial t^4} V_s(x,t) + G \mathcal{R} \{B, H\} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_s(x,t) \right] + \frac{G \mathcal{R} \{B, H\}}{\rho B H} \frac{\partial^2}{\partial t^2} V_s(x,t) = \frac{d^2}{dt^2} \Psi(t) \tag{7}
\]

From equations [6] and [7] it is clear that both \( V_b \) and \( V_s \) contain a function which only depends on the time: \( V_b + f(t) \) and \( V_s + g(t) \).

Rearrangement of the two basic differential equations leads to:

\[
E I \frac{\partial^4}{\partial x^4} V_b(x,t) + \rho B H \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} V_b(x,t) - \rho I \frac{\partial^4}{\partial x^4 \partial t^2} V_b(x,t) + \frac{\rho B H}{G \mathcal{R} \{B, H\}} \left[ \rho I \frac{\partial^4}{\partial t^4} V_b(x,t) - E I \frac{\partial^2}{\partial x^2 \partial t^2} V_b(x,t) \right] = Q(x,t) \tag{8}
\]

\[
I \left[ \frac{\partial^2}{\partial t^2} - \frac{G \mathcal{R} \{B, H\}}{\rho B H} \frac{\partial^2}{\partial x^2} \right] \left[ \rho \frac{\partial^2}{\partial t^2} - E \frac{\partial^2}{\partial x^2} \right] V_s(x,t) + G \mathcal{R} \{B, H\} \frac{\partial^2}{\partial t^2} V_s(x,t) = \frac{I}{\rho B H} \left[ \frac{\partial^2}{\partial t^2} - E \frac{\partial^2}{\partial x^2} \right] Q(x,t) \tag{9}
\]
After estimations for the several coefficients the following two (approximated) differential equations for the pure bending deflection $V_b$ and the shear deflection $V_s$ can be established (see reference 1):

\[
\begin{align*}
E.I \frac{\partial^4}{\partial x^4} V_b(x,t) + \rho B H \frac{\partial^2}{\partial t^2} V_b(x,t) &= + Q(x,t) \\
G \mathfrak{R}(B,H) \frac{\partial^2}{\partial x^2} V_s(x,t) &= - Q(x,t)
\end{align*}
\]  \hspace{1cm} [10a,b]

Equations 10\textsubscript{a} and 10\textsubscript{b} are used for the forward calculation of the deflection $V_{\text{tot}} = V_b + V_s$ if the other parameters like $E$, $B$, $H$, $G$, $\rho$ and $Q(x,t)$ are given. Notice that $Q(x,t)$ has the dimension $N/m$; in equation 10 it is a force per length. The deviation made by the ignoring the other terms in equations 8 and 9 is negligible. **Mark that if extra moving masses are present, the two differential equations are coupled by an extra term (common force) in both equations:**

\[
Q(x_{\text{mass}},t) = F_{\text{mass}}(x_{\text{mass}},t) \delta(x_{\text{mass}}) = M \frac{\partial^2}{\partial t^2} V_t(x_{\text{mass}},t) \delta(x_{\text{mass}}) = M \frac{\partial^2}{\partial t^2} [V_b(x_{\text{mass}},t) + V_s(x_{\text{mass}},t)] \delta(x_{\text{mass}})
\]  \hspace{1cm} [10c]

This expression has the dimension $N/m$ and acts only at $x = x_{\text{mass}}$ and $L_{\text{tot}} - x_{\text{mass}}$ and at $x = \Delta$ and $L_{\text{tot}} - \Delta$. See also equation [12] and [13].

Equations 10\textsubscript{a} and 10\textsubscript{b} will be solved (the steady state) for a sinusoidal load signal of the form: $e^{i\omega t}$.

### 2. BOUNDARY CONDITIONS

In case of a four point bending test (4PB) the following boundary conditions are valid for equal sinusoidal point loads at the two inner clamps and only regarding the steady state condition:

**At** $x = 0$ and $L_{\text{tot}}$:

\[
M(x,t) = - E.I \frac{\partial^2}{\partial x^2} V_b(x,t) = 0 \quad \&
D(x,t) = + G \mathfrak{R}(B,H) \frac{\partial}{\partial x} V_s(x,t) = 0 \quad [I]
\]

**At** $x = \Delta$ and $L_{\text{tot}} - \Delta$:

\[
V_t(x,t) = V_b(x,t) + V_s(x,t) = 0 \quad [II]
\]

**At** $x = L_{\text{tot}}/2$:

\[
\frac{\partial}{\partial x} V_b(x,t) = 0 \quad \& \quad \frac{\partial}{\partial x} V_s(x,t) = 0 \quad [III]
\]  \hspace{1cm} [11a,b,c]
3. DEVELOPMENT OF Q(x,t) IN SERIES OF SINES (orthogonal functions)

For the solution of the differential equations it is necessarily\(^1\) to develop the discrete position functions for the point loads into series of orthogonal functions. Given the simple character of the equation a common sinus or cosines series is already sufficient.

The discrete position function for the load distribution \(Q(x,t)\) is given by:

\[
Q(x,t) = \frac{F_0}{2} [ -\delta(x = \Delta) + \delta(x = \Delta + A) + \delta(L_{tot} - \Delta - A) - \delta(L_{tot} - \Delta) ] e^{i\omega t}
\]  

[12]

in which \(\delta(x=a)\) represents the delta function:

\[
\delta(x=a) = \int_{a^-}^{a^+} f(x) \, dx = f(a)
\]  

[13]

In case of only pure bending a series of orthogonal sine functions would be sufficient. At \(x=0\) and \(x=L_{tot}\) the moment \(M (\propto \partial^2 / \partial x^2)\) has to be zero, which in case of a sine function is automatically fulfilled:

\[
Q(x,t) = F \{ x \} e^{i\omega t} = \sum_{n=1,3,5,\ldots} F_n \sin \left( n \pi \frac{x}{L_{tot}} \right) e^{i\omega t}
\]  

[14]

Because of symmetry only odd numbers for \(n\) are allowed.

If instead of bending only shear deformation occurs, a development in cosine functions is more appropriate because now the shear force \(D (\propto \partial / \partial x)\) has to vanish at \(x=0\) and \(x=L_{tot}\).

\[
Q(x,t) = G \{ x \} = \sum_{n=2,4,6,\ldots} G_n \cos \left( n \pi \frac{x}{L_{tot}} \right) e^{i\omega t}
\]  

[15]

In view of the symmetry now only the even numbers for \(n\) are allowed. Determination of the coefficients \(F_n\) and \(G_n\) is based on the orthogonal property of the sinusoidal functions on the interval from 0 to \(L_{tot}\), which is represented by the following equations:

\[
\begin{align*}
\int_0^{L_{tot}} \sin \left( m \pi \frac{x}{L_{tot}} \right) \sin \left( n \pi \frac{x}{L_{tot}} \right) \, dx &= \begin{cases} 0 & \text{for } n \neq m \\ \frac{L_{tot}}{2} & \text{for } n = m \end{cases} \\
\int_0^{L_{tot}} \cos \left( m \pi \frac{x}{L_{tot}} \right) \cos \left( n \pi \frac{x}{L_{tot}} \right) \, dx &= \begin{cases} 0 & \text{for } n \neq m \\ \frac{L_{tot}}{2} & \text{for } n = m \end{cases}
\end{align*}
\]  

[16]

It should be noted that because of symmetry only odd \(n\) numbers are allowed in case of the development in sine functions and only even \(n\) numbers in case of the development in cosine functions. A combined form (e.g. a Fourier series) is not possible because the orthogonal property (a constant value) of the series doesn’t hold for the product of a sine function and a cosine function. Instead of a constant value (equation 16) the integration depends on \(n\) and \(m\):

---

\(^1\) When no extra moving masses are present the solution of the differential equation 10b which describes the deflection \(V_s\) due to shear can be obtained directly as the product of the time function and the solution for the (pseudo) static case. This last solution can be easily found by integration over the several intervals. This is due to the fact that without a coupling by extra moving masses the phase lag in the deflection \(V_s\) has to be equal to the (opposite) phase lag of the complex stiffness modulus \(S_{mix}\) of the ‘beam’. See also Annex I & II.
\[ \int_0^{L_{\text{tot}}} \sin \left( \frac{m\pi}{L_{\text{tot}}} \right) \cos \left( \frac{n\pi}{L_{\text{tot}}} \right) \, dx = \begin{cases} L_{\text{tot}} \cdot \frac{2m}{\pi} & \text{for } n + m = \text{even} \\ 0 & \text{for } n + m = \text{odd} \end{cases} \]  

The discrete point load functions can now be transformed in either a series of sine functions (pure bending) or in a series of cosine functions (shear deformation):

\[
Q(x,t) = \left\{ \begin{array}{ll}
F(x) = \sum_{n=1}^{\infty} F_n \sin \left( \frac{n\pi}{L_{\text{tot}}} \right) e^{i\alpha n} & \text{with} \\
G(x) = \sum_{n=2}^{\infty} G_n \cos \left( \frac{n\pi}{L_{\text{tot}}} \right) e^{i\alpha n} & \text{where} \\
\end{array} \right.
\]

Furthermore it should be marked that the value \( n=0 \) is not possible for the development in cosine functions (see the equation for the shear deformation; dividing by \( n^2 \)).

Replacing \( F(\chi) \) and \( G(\chi) \) in equation [18] by the expression for \( Q(\chi, t) \) with the delta function \( \delta(\chi) \) leads to equations [19] and [20] for the coefficients \( F_n \) and \( G_n \). The equation for the coefficient \( F_n \) is also given in part II of this series (using the symbol \( A_n \)).

\[
F_n = \frac{F_0}{L_{\text{tot}}} \left[ -\sin \left( \frac{n\pi}{L_{\text{tot}}} \right) + \sin \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) + \sin \left( \frac{n\pi}{L_{\text{tot}}} - A - \Delta \right) - \sin \left( \frac{n\pi}{L_{\text{tot}}} - \Delta \right) \right] \\
G_n = \frac{F_0}{L_{\text{tot}}} \left[ -\cos \left( \frac{n\pi}{L_{\text{tot}}} \right) + \cos \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) + \cos \left( \frac{n\pi}{L_{\text{tot}}} - A - \Delta \right) - \cos \left( \frac{n\pi}{L_{\text{tot}}} - \Delta \right) \right] \\
F_n = 2 \cdot \frac{F_0}{L_{\text{tot}}} \left[ \sin \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) - \sin \left( \frac{n\pi}{L_{\text{tot}}} \right) \right] \text{ for } n = 1,3,5,7 \text{ etc.} \\
G_n = 2 \cdot \frac{F_0}{L_{\text{tot}}} \left[ \cos \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) - \cos \left( \frac{n\pi}{L_{\text{tot}}} \right) \right] \text{ for } n = 2,4,6 \text{ etc.}
\]

In accordance with the formulations used in Part II of this series the following abbreviations are introduced:

\[
T_n(x) = \sin \left( \frac{n\pi}{L_{\text{tot}}} \right) \left[ \sin \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) - \sin \left( \frac{n\pi}{L_{\text{tot}}} \right) \right] \\
U_n(x) = \cos \left( \frac{n\pi}{L_{\text{tot}}} \right) \left[ \cos \left( \frac{n\pi}{L_{\text{tot}}} + \Delta \right) - \cos \left( \frac{n\pi}{L_{\text{tot}}} \right) \right]
\]
If $L_{tot} = L_{eff}$ ($\Delta = 0$) then in accordance with earlier notations in part I of these series the following functions will notations will be used: $T_n(x) \rightarrow P_n(x)$ and $U_n(x) \rightarrow R_n(x).

Mark that for $\Delta = 0$ the coefficient $G_n$ and the function $U_n(x)$ are given by $(n=2,4,6,\ldots)$:

$$G_n = 2 \frac{F_0}{L_{tot}} \left[ \cos \left( n \pi \frac{A}{L_{tot}} \right) - 1 \right] \quad ; \quad U_n(x) \Rightarrow R_n(x) = \cos \left( n \pi \frac{x}{L_{tot}} \right) \left[ \cos \left( n \pi \frac{A}{L_{tot}} \right) - 1 \right]$$

In this way the point loads at the inner and outer supports can be taken into account by transforming those four point loads in a force distribution $Q\{x,t\}$ along the beam. The force distribution $Q\{x,t\}$ can be either represented as

$$Q\{x,t\} = 2 \frac{F_0}{L_{tot}} \sum_{n=1}^{\infty} \left[ T_n(x) \right] e^{i n \omega t} \quad (n = \text{odd}) \quad \text{or as} \quad Q\{x,t\} = 2 \frac{F_0}{L_{tot}} \sum_{n=1}^{\infty} \left[ U_n(x) \right] e^{i n \omega t} \quad (n = \text{even})$$

\[23\]
4. DEVELOPMENT OF THE (OTHER) MASS FORCES IN SERIES

First of all it should be mentioned that the placing of a single extra mass at an arbitrarily chosen location (except \( x = \frac{L_{tot}}{2} \)) will not be dealt with. Only symmetrically placed masses will be taken into account. Normally these extra masses are located at the inner clamps (mass of the plunger etc.). These extra forces will like the external driven force raise reaction forces at the outer clamps. If the mass forces are denoted by \( F_{mass} \) at \( x_{mass} \) and \( L_{tot} - x_{mass} \), the force distribution \( Q_{mass}(x_{mass}, t) \) will be given by:

\[
Q_{mass}(x_{mass}, t) = \sum_{n=2,4}^{\infty} \left[ \frac{2}{L_{tot}} \int_0^L F(\xi) \sin \left( n\pi \frac{x_{mass}}{L_{tot}} \right) d\xi \right] \sin \left( n\pi \frac{x}{L_{tot}} \right) e^{i\omega t} = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ \sin \left( n\pi \frac{x_{mass}}{L_{tot}} \right) \sin \left( n\pi \frac{\Delta}{L_{tot}} \right) \right] e^{i\omega t}
\]

for implementation in the pure bending equation (10_a) and for the shear equation (10_b) by:

\[
Q_{mass}(x_{mass}, t) = \sum_{n=2,4}^{\infty} \left[ \frac{2}{L_{tot}} \int_0^L F(\xi) \cos \left( n\pi \frac{x_{mass}}{L_{tot}} \right) d\xi \right] \cos \left( n\pi \frac{x}{L_{tot}} \right) e^{i\omega t} = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ \cos \left( n\pi \frac{x_{mass}}{L_{tot}} \right) \cos \left( n\pi \frac{\Delta}{L_{tot}} \right) \right] e^{i\omega t}
\]

Notice the difference with the expression for the (driven) force distribution \( Q[x, t] \), if \( x_{mass} \) is not equal to the location of the inner clamp: \( A + \Delta \). To get in line with the expression for this force distribution \( Q[x, t] \) the following coefficients are introduced:

\[
T_n^+(x_{mass}) = \left[ \frac{\sin \left( n\pi \frac{x_{mass}}{L_{tot}} \right) - \sin \left( n\pi \frac{\Delta}{L_{tot}} \right)}{\sin \left( n\pi \frac{A + \Delta}{L_{tot}} \right) - \sin \left( n\pi \frac{\Delta}{L_{tot}} \right)} \right]
\]

\[
Q_{mass}(x_{mass}, t) = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ \sin \left( n\pi \frac{x_{mass}}{L_{tot}} \right) \left[ \sin \left( n\pi \frac{x_{mass}}{L_{tot}} \right) - \sin \left( n\pi \frac{\Delta}{L_{tot}} \right) \right] \right] e^{i\omega t} = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ T_n^+(x_{mass}) T_n(x) \right]
\]

\[
U_n^+(x_{mass}) = \left[ \frac{\cos \left( n\pi \frac{x_{mass}}{L_{tot}} \right) - \cos \left( n\pi \frac{\Delta}{L_{tot}} \right)}{\cos \left( n\pi \frac{A + \Delta}{L_{tot}} \right) - \cos \left( n\pi \frac{\Delta}{L_{tot}} \right)} \right]
\]

\[
Q_{mass}(x_{mass}, t) = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ \cos \left( n\pi \frac{x_{mass}}{L_{tot}} \right) \left[ \cos \left( n\pi \frac{x_{mass}}{L_{tot}} \right) - \cos \left( n\pi \frac{\Delta}{L_{tot}} \right) \right] \right] e^{i\omega t} = \frac{2F_{mass}}{L_{tot}} \sum_{n=2,4}^{\infty} \left[ U_n^+(x_{mass}) U_n(x) \right]
\]

\[27_a,b\]
Expression for the forces due to moving masses at the inner clamps

\[
Q_{\text{clamp}}(x,t) = + M_{\text{clamp}} V_t \{x = \Delta + A\} \cdot \frac{\partial^2}{\partial t^2} e^{i\omega_0 t + i\theta_{\text{clamp}}} = - M_{\text{clamp}} V_t \{x = \Delta + A\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{clamp}}}
\]

\[
= - M_{\text{clamp}} V_t \{x = \Delta + A\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{clamp}}} \cdot \left( \frac{2}{L_{\text{tot}}} \cdot \sum_{n=1}^{\infty} [T_n(x)] \right)
\]

or

\[
= - M_{\text{clamp}} V_t \{x = \Delta + A\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{clamp}}} \cdot \left( \frac{2}{L_{\text{tot}}} \cdot \sum_{n=2}^{\infty} [U_n(x)] \right)
\]

[28]

If \( x_{\text{mass}} \) equals \( A + \Delta \) than of course \( T_n^*\{x_{\text{mass}}\} = U_n^*\{x_{\text{mass}}\} = 1. \)

Expression for the force due to a moving mass at the centre

\[
Q_{\text{center}}(x,t) = + M_{\text{center}} V_t \{x = x_{\text{center}}\} \cdot \frac{\partial^2}{\partial t^2} e^{i\omega_0 t + i\theta_{\text{center}}} = - M_{\text{center}} V_t \{x = x_{\text{center}}\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{center}}}
\]

\[
= - M_{\text{center}} V_t \{x = x_{\text{center}}\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{center}}} \cdot \left( \frac{2}{L_{\text{tot}}} \cdot \sum_{n=1}^{\infty} [T_n(x)] \cdot T_n(x) \right)
\]

or

\[
= - M_{\text{center}} V_t \{x = x_{\text{center}}\} \cdot \omega_0^2 e^{i\omega_0 t + i\theta_{\text{center}}} \cdot \left( \frac{2}{L_{\text{tot}}} \cdot \sum_{n=2}^{\infty} [U_n(x)] \cdot U_n(x) \right)
\]

[29]
5. FUNCTION FORMULATION FOR Vb and Vs

When extra moving masses are included the original differential equations 10, have to be rewritten as:

\[
E.I \cdot \frac{\partial^4}{\partial x^4} V_b(x,t) + \rho.B.H \cdot \frac{\partial^2}{\partial t^2} V_b(x,t) = + Q(x,t) - Q_{\text{mass}}(x,t)
\]  

\[
G.R(B,H) \cdot \frac{\partial^2}{\partial x^2} V_s(x,t) = - Q(x,t) + Q_{\text{mass}}(x,t)
\]  

The load distributions along the beam are given by:

\[
Q(x,t) = 2 \frac{F_0}{L_{\text{tot}}} \sum_{n=1}^{\infty} [T_n(x)] \text{e}^{i \omega_n t} \quad \text{(n = odd)} \quad \text{or as} \quad Q(x,t) = 2 \frac{F_0}{L_{\text{tot}}} \sum_{n=2}^{\infty} [U_n(x)] \text{e}^{i \omega_n t} \quad \text{(n = even)}
\]  

\[
Q_{\text{mass}}(x_{\text{mass}},t) = 2 \frac{F_{\text{mass}}}{L_{\text{tot}}} \sum_{n=1,3}^{\infty} [T_n(x_{\text{mass}})T_n(x)] \quad \text{or as} \quad Q_{\text{mass}}(x_{\text{mass}},t) = 2 \frac{F_{\text{mass}}}{L_{\text{tot}}} \sum_{n=2,4}^{\infty} [U_n(x_{\text{mass}})U_n(x)]
\]  

The parameter \( F_{\text{mass}} \) is given by:

\[
F_{\text{mass}} = M_{\text{mass}} \frac{d^2}{dt^2} V_t(x_{\text{mass}},t) = M_{\text{mass}} \frac{d^2}{dt^2} [V_b(x_{\text{mass}},t) + V_s(x_{\text{mass}},t)] = - M_{\text{mass}} \omega_0^2 V_t(x_{\text{mass}},t)
\]  

All forces are written in the form of sine or cosine series using \( T_n(x) \) or \( U_n(x) \). Therefore it is logical to assume the following expressions for \( V_b(x) \) and \( V_s(x) \):

\[
V_b(x,t) = [V_a(x) + V_c(x) + V_d(x)] \text{e}^{i \omega t} = \left[ \sum_{n=1}^{\infty} (A_n e^{-i \omega_n} T_n(x)) + V_c(x) + V_d(x) \right] \text{e}^{i \omega t}
\]

\[
V_s(x,t) = [V_g(x) + H_0] \text{e}^{i \omega t} = \left[ \sum_{n=1}^{\infty} (H_n e^{-i \omega_n} U_n(x)) + H_0 \right] \text{e}^{i \omega t}
\]

The function \( V_a(x) \) satisfies the complete differential equation including the forces induced by extra moving masses.

The functions \( V_c(x) \) and \( V_d(x) \) are solutions of the “homogenous” differential equations without the external force and induced forces. They are needed to satisfy the boundary requirements at the outer supports as will be shown later on.

The function \( V_g(x) \) plays the same role as \( V_a(x) \) in the differential equation for the shear deformation.

The constant \( H_0 \) (being a constant because at \( x = 0 \) the shear force must be zero) is comparable in function to \( V_c(x) \) and \( V_d(x) \) and is also needed because at the outer clamp the total deflection \( V_t(\Delta t) = V_b(\Delta t) + V_s(\Delta t) \) has to vanish.

The deflection \( V_b(\Delta t) \) equals zero because \( V_a(\Delta) + V_c(\Delta) + V_d(\Delta) = 0 \).
The functions $V_c(x)$ and $V_d(x)$ are coupled to each other because they have to meet (together) the requirement for the bending moment of $V_b(x)$ at $x=0$ and $x=L$.

Also $V_s(\Delta t)$ has to be zero, which is accomplished by $V_g(\Delta) + H_0 = 0$.

Remember that $V_b(\Delta, t) = V_b(L_{tot} - \Delta, t)$ and $V_s(\Delta, t) = V_s(L_{tot} - \Delta, t)$.

6. **THEORETICAL SOLUTION WHEN NO EXTRA MASSES ARE PRESENT**

In this case the two differential equations are not coupled by an extra moving mass at an arbitrarily location. For a viscous-elastic material the equations can be written as:

\[
S_{mix} e^{i\phi} I \frac{\partial^4}{\partial x^4} V_b(x,t) + \rho B H \frac{\partial^2}{\partial t^2} V_b(x,t) = + Q(x,t) = \frac{2F_0}{L_{tot}} \sum_{n=1}^{\infty} [T_{2n-1}(x)] e^{i\omega t}
\]

\[
\frac{S_{mix}}{2(1 + \nu)} e^{i\phi} (\alpha B H) \frac{\partial^2}{\partial x^2} V_s(x,t) = - Q(x,t) = - \frac{2F_0}{L_{tot}} \sum_{n=1}^{\infty} [U_{2n}(x)] e^{i\omega t}; \text{ Note: } \nu \text{ is taken Real }
\]

\[
V_b(x,t) = V_b(x)e^{i\omega t}; \quad V_s(x,t) = V_s(x)e^{i\omega t}
\]

**Bending Deflection $V_b$**

The solution for $V_b$ is now represented by ($\rho B H L_{tot} = M_{beam}$):

\[
V_b(x) = V_a(x) + V_c(x) + V_d(x) = \sum_{n=1}^{\infty} [V_{a,2n-1}(x)] + V_c(x) + V_d(x) = \sum_{n=1}^{\infty} [A_{2n-1} e^{-i\varphi_{2n-1}} T_{2n-1}(x)] + V_c(x) + V_d(x)
\]

With

\[
A_{2n-1} e^{-i\varphi_{2n-1}} = \frac{2F_0}{S_{mix} e^{i\phi} I \left(\frac{2n-1}{4}\right) \pi^2 - M_{beam} \omega^2}
\]

The similarity with a mass-spring system is already obvious. Later on the expressions will be given which are used in the Excel program. For the complete solution the function $V_c(x)$ and $V_d(x)$ are needed when the total length of the beam is bigger than the distance between the two outer clamps:

\[
V_c(x) = \sum_{n=1}^{\infty} C_{2n-1} \cos \left( \beta_n \left( x - \frac{L_{tot}}{2} \right) e^{-i\varphi \frac{x}{4}} \right)
\]

The function $V_c(x)$ satisfies the homogenous differential equation (without the force distribution $Q(x)$). Because the differential equation is of the order 4, the complementary function $V_c(ix)$ is also a solution ($i^4 = +1$).

\[
V_d(x) = \sum_{n=1}^{\infty} D_{2n-1} \cos \left( i \beta_n \left( x - \frac{L_{tot}}{2} \right) e^{-i\varphi \frac{x}{4}} \right) = \sum_{n=1}^{\infty} D_{2n-1} \cosh \left( \beta_n \left( x - \frac{L_{tot}}{2} \right) e^{-i\varphi \frac{x}{4}} \right); \text{ with } \beta_n = \sqrt{\frac{M_{beam} \omega^2}{S_{mix} L_{tot}}}
\]
Finally the following requirements have to be fulfilled:

**Requirements :**

\[ \frac{\partial^2}{\partial x^2} \{ V_a(x) + V_c(x) + V_d(x) \} + V_a x V_c x V_d \equiv 0 \] \[ \text{at } x=0 \]  \[ \Rightarrow \frac{\partial^2}{\partial x^2} V_a(x) |_{x=0} = 0 \] \[ \Rightarrow D_{2n-1} = \frac{\cos \left( \frac{L_{\text{tot}} \beta_0}{2} \right)}{\cosh \left( \frac{L_{\text{tot}} \beta_0}{2} \right)} C_{2n-1} = \lambda_0 C_{2n-1} \]

**Shear Deflection Vs**

**Important:** As mentioned before, due to the lack of extra moving masses the solution can be obtained as the product of the time function \( e^{i \omega t} \) and the static solution. The last one can be established by a simple (double) integration of the differential equation in which the point loads are represented as the product of the load and a Dirac function \( \delta(x) \). However, when a coupling exists between the two differential equations this solution procedure will fail. In that case a development in sines or cosines has to be used as done in this chapter. See also Annex I and II.

Given the fact that the first derivative has to be zero at \( x = 0 \) and at \( x = L_{\text{tot}} \) the deflection \( Vs \) is chosen as \(^{2}\):

\[ Vs(x) = \sum_{n=1}^{\infty} \left[ V_{e,2n}(x) \right] = \sum_{n=1}^{\infty} \left[ H_{2n} e^{-i \varphi_{2n}} U_{2n}(x) \right] + H_0 \]

\[ H_{2n} e^{-i \varphi_{2n}} = \frac{2 F_0}{L_{\text{tot}}} \cdot \frac{2(1 + \nu)}{S_{\text{mix}} e^{i \alpha B} H} \cdot \frac{L_{\text{tot}}^2}{(2n)^2 \pi^2} ; \text{ Notice that } \varphi_{2n} = \varphi \]

**Requirement**

\[ Vs \{ \Delta \} = 0 \Rightarrow H_0 = -\sum_{n=1}^{\infty} \left[ H_{2n} e^{-i \varphi_{2n}} U_{2n}(\Delta) \right] \]

\(^{2}\) In this case the load distribution is developed into a cosine series. Therefore the requirement that the shear force at \( x=0 \) must be zero is automatically fulfilled. By adding a constant the requirement that the deflection has to be zero for \( x=\Delta \) can be met.
It is also possible to obtain an alternative solution expression for $V_s^3$:

Alternative: $V_s(x) = \sum_{n=1}^{\infty} \left[ V_{g,2n-1}^+(x) \right] = \sum_{n=1}^{\infty} \left[ H_{2n-1}^+ e^{-i\varphi_{2n-1}} T_{2n-1} \{x\} \right] + H_0^+ \frac{x}{L_{tot}} \ ; \ 0 < x < \frac{L_{tot}}{2}$

$$H_{2n-1}^+ e^{-i\varphi_{2n-1}} = \frac{2F_0}{L_{tot}} \cdot \frac{2(1+\nu)}{S_{mix} e^{i\varphi} \alpha B H} \cdot \frac{L_{tot}^2}{(2n-1)^3 \pi^2} \ ; \ \text{Notice that } \varphi_{2n-1} = \varphi$$

$$\frac{\partial}{\partial x} \sum_{n=1}^{\infty} \left[ V_{g,2n-1}^+(x) \right] \bigg|_{x=0} + \frac{H_1^+}{L_{tot}} \Delta + H_0^+ + \sum_{n=1}^{\infty} \left[ H_{2n-1}^+ e^{-i\varphi_{2n-1}} T_{2n-1} \{\Delta\} \right] = 0$$

$$H_i^+ = \sum_{n=1}^{\infty} H_{1,2n-1}^+ \Rightarrow H_i^+ = - \frac{2F_0}{L_{tot}} \cdot \frac{2(1+\nu)}{S_{mix} e^{i\varphi} \alpha B H} \cdot \frac{L_{tot}^2}{(2n-1)^3 \pi^2} \frac{(2n-1)\pi}{L_{tot}} T_{2n-1} \{\pi/2\}$$

$$H_0^+ = \sum_{n=1}^{\infty} H_{0,2n-1}^+ \Rightarrow H_0^+ = \left[ \frac{2F_0}{L_{tot}} \cdot \frac{2(1+\nu)}{S_{mix} e^{i\varphi} \alpha B H} \cdot \frac{L_{tot}^2}{(2n-1)^3 \pi^2} \right] \left[ T_{2n-1} \{\Delta\} + \Delta T_{2n-1} \{0\} \right]$$

[42]

The alternative solution is expressed in the odd $T_{2n-1}\{x\}$ series used in the solution for pure bending and could be from this point an attractive alternative. However, this solution require much more terms compared to the basic solution using the even $U_{2n}\{x\}$ series$^d$.

Calculation of Shear Deflection

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</thead>
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</tr>
<tr>
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</tr>
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</table>

Figure 1  The development of the Shear Deflection using a Cosine ($U_n$) and a Sine ($T_n$) series

$^d$ When the load distribution is developed into a sine series, two solutions for the homogenous differential equation are needed in order to fulfill both the requirements at $x=0$ and $x=\Delta$. Furthermore the derivate of $T_n\{x\}$ at $x=0$ equals $2n\pi/L$. $T_n\{\pi/2\}$

$^d$ If $\Delta = 0$ the $T_{2n-1}$ series alone do not satisfy the requirement for the shear force at $x = 0$. The $T_{2n-1}$ series will lead after derivation to a series of $1/(2n-1)$ terms. In order to satisfy the boundary restriction a solution of the homogenous differential equation has to be added: $-x/(2n-1)$. This solution will eliminate the shear force after the derivation of the terms $-x/(2n-1)$ at $x = 0$. However, the series $1/(2n-1)$ converges badly.
It is also possible to express $V_s$ in $V_b$ using equation $1_a$ or equation $4$. However equation $4$ contains an unknown function in $t$ and equation $1_a$ is in fact similar to equation $10_b$ because the first term in equation $1_a$ can be neglected compared to the second term. Therefore in the Excel program only the final solution for $V_t$ is obtained by developing $V_s$ in cosine series.

Total Length $L_{tot}$ equals the Effective Length $L_{eff}$ ($\Delta=0$)

In that case the basic series development $T_n$ and $U_n$ are defined by the symbol $P_n$ and $R_n$:

If $L_{tot} = L_{eff}$ ($\Delta=0$) than:

$$T_{2n-1}(x) \Rightarrow P_{2n-1}(x) = \sin \left( (2n-1) \pi \frac{x}{L_{eff}} \right) \cdot \sin \left( (2n-1) \pi \frac{A}{L_{eff}} \right) ; \quad P_{2n-1}(0) = 0$$

$$U_{2n}(x) \Rightarrow R_{2n}(x) = \cos \left( (2n) \pi \frac{x}{L_{eff}} \right) \cdot \cos \left( (2n) \pi \frac{A}{L_{eff}} \right) - 1 ; \quad R_{2n}(0) = \cos \left( (2n) \pi \frac{A}{L_{eff}} \right) - 1$$

$$V_b(x) = \sum_{n=1}^{\infty} [V_s,2n-1] \cdot A_{2n-1} e^{-i \phi_{n-1}} P_{2n-1}(x) ; \quad V_c(x) = V_d(x) = 0$$

$$V_s(x) = \sum_{n=1}^{\infty} [V_s,2n] \cdot H_{2n} e^{-i \phi_{2n}} R_{2n}(x) + H_0 ; \quad H_0 = -\sum_{n=1}^{\infty} H_{2n} e^{-i \phi_{2n}} \left[ \cos \left( (2n) \pi \frac{A}{L_{eff}} \right) - 1 \right]$$

7. SOLUTION FOR THE PSEUDO-STATIC CASE

In the pseudo-static case the inertia forces do not play a role. Therefore the ratio of $V_s$ and $V_b$ can easily be determined from equation [4] by omitting the time dependent terms.

$$V_s \{x\} = -\frac{I E \cdot \frac{d^2}{dx^2} V_b \{x\}}{G \cdot \Re \{B, H\}} \quad \text{In which } E \text{ is the Stiffness of the beam} \quad [44]$$

For $(A+\Delta) < x < \frac{L_{tot}}{2}$ the bending deflection $V_b$ is:

$$V_b = \frac{F_0 \left( L_{tot} - 2 \Delta \right)^3}{12 \cdot E \cdot I} \cdot \frac{A}{(L_{tot} - 2 \Delta) \cdot \left( 3 \cdot \frac{x - \Delta}{L_{tot} - 2 \Delta} - 3 \left( \frac{\Delta}{L_{tot} - 2 \Delta} \right)^2 - \left( \frac{A}{L_{tot} - 2 \Delta} \right)^2 \right)} \quad [45]$$

$$V_s = -\frac{I E \cdot \frac{d^2}{dx^2} V_b \{x\}}{G \cdot \Re \{B, H\}} = \frac{F_0 A (1 + \mu)}{E \cdot \Re \{B, H\}} ; \quad V_b \left\{ \frac{L_{tot}}{2} \right\} = \frac{F_0 A \left[ 3 (L_{tot} - 2 \Delta)^2 - 4 A^2 \right]}{12 \cdot E \cdot B \cdot H^3}$$

$$\Rightarrow V_s \{(A+\Delta) < x < \frac{L_{tot}}{2}\} = + \frac{F_0 A}{2 \cdot G \cdot \Re \{B, H\}} \quad \Rightarrow V_b \left\{ \frac{L_{tot}}{2} \right\} = \frac{4(1 + \mu) \cdot H^2}{\alpha \cdot 3 \left( L_{tot} - 2 \Delta \right)^2 - 4 A^2} \quad [46]$$

5 In ASTM standards $A=L/3$; Using $\alpha = 2/3$ the ratio will be:

$$\frac{V_s \left\{ \frac{L}{2} \right\}}{V_b \left\{ \frac{L}{2} \right\}} = \frac{544}{23} \cdot \left( 1 + \mu \right) \left( \frac{H}{L} \right)^2$$
For $\Delta < x < A + \Delta$ the bending deflection $V_b$ is:

$$V_b = \frac{F_0 (L_{tot} - 2\Delta)^3}{12EI} \left[ \frac{(x - \Delta)}{(L_{tot} - 2\Delta)} \cdot 3 \left( \frac{A}{L_{tot} - 2\Delta} \right)^2 - \left( \frac{x - \Delta}{L_{tot} - 2\Delta} \right)^2 \right]$$

$$\Rightarrow V_s \{ \Delta < x < A + \Delta \} = - \frac{I \cdot E \cdot \frac{d^2}{dx^2} V_b \{ x \}}{G\Re \{ \alpha, \beta \}} = \frac{F_0 (1 + \mu)}{\alpha E} \cdot (x - \Delta)$$

[48]

For $0 < x < \Delta$ the bending deflection $V_b$ is:

$$V_b = \frac{F_0 (L_{tot} - 2\Delta - A)}{4E I} A \left( x - \Delta \right) \Rightarrow V_s \{ 0 < x < \Delta \} = 0$$

[49]

In many textbooks and papers the value for $\alpha$ is taken equal to $2/3$. However, based on 1D, 2D and 3D finite element calculations with ABAQUS it was found that a value of 0.85 was more appropriate. That’s to say the answers from 3D (and 1D) calculations are comparable to the analytical solutions if a value of 0.85 was used. As expected the ratio depends on 3 parameters: Poisson ratio $\mu$, the ratio $H/L$ and the ratio $A/L$.

The value $\alpha$ is obtained by comparing FEM calculations for the 1D case (bar elements) and for the 2D case. In the 1D case it is possible to vary the stiffness modulus $E$ and the shear modulus $G$ independently. In this way the deflections due to pure bending and shear can be determined separately (by choosing an “infinite” value for $G$). Assuming that the 2D case (and 3D case) will give correct answers the ratio $V_s \{ L_{tot}/2 \}/V_b \{ L_{tot}/2 \}$ is determined. For $H/L_{tot}$ ratio of 25/450; 50/450 and 100/450 an $\alpha$ value of 0.85 was established.

8. HOW TO INCORPORATE EXTRA MOVING MASSES.

The extra inertia forces are present in both differential equations:

$$S_{mix} e^{i\psi} \frac{\partial^4}{\partial x^4} V_b(x,t) + \rho B H \frac{\partial^2}{\partial t^2} V_b(x,t) + M_{max} \frac{\partial^2}{\partial t^2} V_t(x_{max},t) = + Q(x,t)$$

$$\frac{S_{mix}}{2(1 + \nu)} e^{i\psi} (\alpha B H) \frac{\partial^2}{\partial x^2} V_s(x,t) - M_{max} \frac{\partial^2}{\partial t^2} V_t(x_{max},t) = - Q(x,t)$$

[50a,b]

$$V_b(x,t) = V_b(x) e^{i\omega t} \; ; \; V_s(x,t) = V_s(x) e^{i\omega t} \; ; \; V_t(x,t) = V_t(x,t) + V_s(x,t)$$

In this report a correct solution procedure will be followed in contrast with a procedure outlined in Part II “Overhanging Beam Ends and Extra Moving Masses” of these series. In Part II the deflection due to shear was ignored and $V_t(x_{max},t)$ was taken equal to $V_b(x_{max},t)$. By multiple iteration (calculating $V_b(x_{max},t)$ and $V_b(x,t)$ the ultimate deflection values were established. This procedure was and is attractive because for the back calculation procedure the extra moving masses could be incorporate in an easy way.

However, if the deflection due to shear is not ignored a different forward calculation has to be performed. First of all the equations 50a,b are rewritten as:
In this way the forces due to an extra moving mass at a prescribed location are included in
the external (driven) force. In fact the same type of formulas as presented in chapter 6 can
be used if the force \( F \) is replaced by: \( F \Rightarrow F - M_{\text{mass}} \alpha_0^2 \text{V}_t{x_{\text{mass}}} \) according to equations [31] to [33]. Because the influences of extra masses are for normal conditions small, a simple
iteration procedure can be performed:

1. The first step is the calculation of the bending and shear deflections at the desired
   location \( x \) and the location(s) \( x_{\text{mass}} \) where the extra moving masses are using the
equations for the system without extra moving masses.

2. Than the total deflection at \( x_{\text{mass}} \) is calculated: \( V_t{x_{\text{mass}}} = V_b{x_{\text{mass}}} + V_s{x_{\text{mass}}} \)

3. The second step is the calculation of the deflections with extra moving masses in
   which for the deflection \( V_t{x_{\text{mass}}} \) at the right hand side of equations [51a,b] the value
   of the former step is taken.

4. Step 3 is repeated until the estimate value for \( V_t{x_{\text{mass}}} \) of the former step doesn’t
differ from the new estimated value for \( V_t{x_{\text{mass}}} \)

5. In the Excel program the iterations are not performed using e.g. Visual Basic.
   Therefore the number of iterations is limited to nine steps which are in normally
   situations enough.
ANNEX I “Development in Sine or Cosine Series”

In chapter 5 the load function $Q$ was developed into infinite a series of sine or cosine functions. With respect to the boundary conditions, the sine function series was adopted for the deflection due to pure bending and the cosine function series was adopted for the deflection due to shear. In this annex we will try the opposite.

**Pure Bending**

When a cosine function series is used the boundary condition for the second derivation is not met. Therefore even in the case of no overhanging beam ends the solutions $V_c(x)$ and $V_d(x)$ have to be added

$$\frac{\partial^2}{\partial x^2} V_b(x,t) \bigg|_{x=0} = \frac{\partial^2}{\partial x^2} \left( \sum_{n=2,4,6,...} \left( A_n e^{-imx} U_n(x) \right) + V_c(x) + V_d(x) \right) e^{im\omega x} \bigg|_{x=0} = 0 \quad I-1$$

The other condition is that at the outer clamp the deflection should be zero.

$$V_b(\Delta,t) = [V_a(\Delta) + V_c(\Delta) + V_d(\Delta)] e^{i\omega t} = \left[ \sum_{n=1}^{\infty} \left( A_{2n} e^{-im\omega \Delta} U_{2n}(\Delta) \right) + V_c(\Delta) + V_d(\Delta) \right] e^{i\omega t} = 0 \quad I-2$$

Therefore we have two equations, which make it possible to calculate the coefficients $C_n$ and $D_n$ of the deflections functions $V_c(x)$ and $V_d(x)$.

**Pure Shear**

In case of the deflection due to shear it is also possible to solve the problem using sine functions. However, the only acceptable solution of the homogenous differential equation for the whole interval ($0 \leq x \leq L_{tot}$) is a single constant $H_0$ times the time function. This is not enough to satisfy both conditions: no shear at $x=0$ and no deflection at $x=\Delta$. However, an acceptable solution will be possible if the interval $0 \leq x \leq L_{tot}$ is divided into 2 separate intervals: $0 \leq x \leq L_{tot}/2$ and $L_{tot}/2 \leq x \leq L_{tot}$. In the first interval we have the extra solution $H_1 x + H_0$ and in the second interval the solution $-H_1 x + H_0 + H_1 L_{tot}$.

$$T_n(x) = \sin \left( (2m-1) \pi \frac{x}{L_{tot}} \right) \left[ \sin \left( (2m-1) \pi \frac{A + \Delta}{L_{tot}} \right) - \sin \left( (2m-1) \pi \frac{\Delta}{L_{tot}} \right) \right]$$

$$\frac{\partial}{\partial x} T_n(x) \bigg|_{x=0} = \frac{(2m-1)\pi}{L_{tot}} \cos \left( (2m-1) \pi \frac{x}{L_{tot}} \right) \left[ \sin \left( (2m-1) \pi \frac{A + \Delta}{L_{tot}} \right) - \sin \left( (2m-1) \pi \frac{\Delta}{L_{tot}} \right) \right]$$

$$\rightarrow At \ x = 0 : \frac{\partial}{\partial x} T_n(x) \bigg|_{x=0} = \frac{(2m-1) \pi}{L_{tot}} \left[ \sin \left( (2m-1) \pi \frac{A + \Delta}{L_{tot}} \right) - \sin \left( (2m-1) \pi \frac{\Delta}{L_{tot}} \right) \right] \quad I-3$$

$$\rightarrow At \ x = L_{tot} : \frac{\partial}{\partial x} T_n(x) \bigg|_{x=L_{tot}} = -\frac{(2m-1) \pi}{L_{tot}} \left[ \sin \left( (2m-1) \pi \frac{A + \Delta}{L_{tot}} \right) - \sin \left( (2m-1) \pi \frac{\Delta}{L_{tot}} \right) \right] \quad I-4$$
\[
\begin{align*}
\alpha &= \sum_{m=1}^{\infty} \left[ \frac{(2m-1)\pi}{L_{\text{tot}}} \left( \sin \left( (2m-1)\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \sin \left( (2m-1)\pi \frac{\Delta}{L_{\text{tot}}} \right) \right) \right] \\
\beta &= -\alpha \Delta - \sum_{m=1}^{\infty} \left[ \sin \left( (2m-1)\pi \frac{\Delta}{L_{\text{tot}}} \right) \left( \sin \left( (2m-1)\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \sin \left( (2m-1)\pi \frac{\Delta}{L_{\text{tot}}} \right) \right) \right] \\
\gamma &= \alpha (L_{\text{tot}} - \Delta) - \sum_{m=1}^{\infty} \left[ \sin \left( (2m-1)\pi \frac{\Delta}{L_{\text{tot}}} \right) \left( \sin \left( (2m-1)\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \sin \left( (2m-1)\pi \frac{\Delta}{L_{\text{tot}}} \right) \right) \right]
\end{align*}
\]

At \( x = L_{\text{tot}}/2 \) the deflection is continuous but the first derivative changes from sign.

**ANNEX II “The series development for the deflection due to shear”**

**INTRODUCTION**

In the “Handbook of Mathematical Functions” (M. Abramowitz & I.A. Segun; Dover publications, Inc., New York) an overview is given of summable series consisting of either sines or cosines.

\[
\sum_{n=1}^{\infty} \frac{\cos(n\varphi)}{n} = -\varphi \ln \left( 2 \sin \left( \frac{\varphi}{2} \right) \right) \quad \text{for } 0 < \varphi < 2\pi
\]

\[
\sum_{n=1}^{\infty} \frac{\cos(n\varphi)}{n^2} = \frac{\pi^2}{6} - \frac{\varphi^2}{2} + \frac{\varphi^2}{4} \quad \text{for } 0 \leq \varphi \leq 2\pi
\]

\[
\sum_{n=1}^{\infty} \frac{\cos(n\varphi)}{n^4} = \frac{\pi^4}{90} - \frac{\varphi^4}{12} + \frac{\varphi^3}{12} - \frac{\varphi^4}{48} \quad \text{for } 0 \leq \varphi \leq 2\pi
\]

\[
\sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n} = \frac{1}{2} \left( \pi - \varphi \right) \quad \text{for } 0 < \varphi < 2\pi
\]

\[
\sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n^2} = \frac{\varphi^2}{6} - \frac{\varphi^3}{4} + \frac{\varphi^3}{12} \quad \text{for } 0 \leq \varphi \leq 2\pi
\]

\[
\sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n^4} = \frac{\varphi^4}{90} - \frac{\varphi^6}{36} + \frac{\varphi^4}{48} - \frac{\varphi^5}{240} \quad \text{for } 0 \leq \varphi \leq 2\pi
\]

The series \( \sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n^2} \) will converge for \( 0 \leq \varphi \leq \pi \) but is not summable like the sine and cosine series given above. The Clausen’s integral can only be expressed in another series representation:

\[
\sum_{n=1}^{\infty} \frac{\sin(n\varphi)}{n^2} = -\varphi \ln |\varphi| + \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1}}{(2n)!} B_{2n} \frac{\varphi^{2n+1}}{2n(2n+1)} \right] \quad \text{for } 0 \leq \varphi < \frac{\pi}{2}
\]

\[ 6 \] The coefficient \( B_{2n} \) is not given in the “Handbook of Mathematical Functions” but can (probably) found in “Tabulations of the function \( \psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} \)” by A. Ashour and A. Sabri (Math. Tables Aids Comp., 1956)
This does not explain the large difference in the convergence of the two solutions (sine or cosine series) for the deflection due to shear. The series expansion for the deflections due to shear and pure bending consist out of a product of two sines or two cosines. Such a product can be reduced to a summation of subtraction of two cosine terms:

\[
\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2} \quad \& \quad \cos(\alpha)\cos(\beta) = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}
\]

The obtained cosine expansions can be summed up due to the term \(n^2\) in the denominator of the cosine terms (eq. II-1). In the next paragraph it will be shown that both series expansions (sine & cosine) will lead to the same answer. Notice that the same conclusion yields for the deflection due to pure bending in which the denominator of the terms contains a term \(n^4\) (eq. II-1). Further it is noticed the series expansion for the strain due to pure bending (2\(^{nd}\) derivation of the deflection \(V_b\) with respect to \(x\)) is build up out of cosines terms divided by \(n^2\) (like the expansion for the shear deflection) and therefore converge fast. This is in contrast with the (bending) “strain” derived in the same way for the shear deflection. The expansion leads in that procedure to a summation of cosine terms without a denominator which depends on \(n\).

**Shear Deflection and “Shear Strain” using the Cosine and Sine series expansion**

**Deflection**

If no extra moving masses are involved the deflection due to shear can be easily obtained from equation [44] for the static case which leads for the interval \(A + \Delta < x < L_{tot}/2\) to:

\[
V_b[x] = \frac{F_0 (L_{tot} - 2\Delta)^3}{12 E I} \cdot \frac{A}{(L_{tot} - 2\Delta)} \cdot \left[ 3 \frac{x - \Delta}{L_{tot} - 2\Delta} - 3 \left( \frac{x - \Delta}{L_{tot} - 2\Delta} \right)^2 - \left( \frac{A}{L_{tot} - 2\Delta} \right)^2 \right]
\]

\[
V_s[x] = -\frac{I E \frac{d^2}{dx^2} V_b[x]}{G \mathcal{R}\{B, H\}} = \frac{F_0 A (1 + \mu)}{E \mathcal{R}\{B, H\}} \cdot \left[ V_b \left( \frac{L_{tot}}{2} \right) = \frac{F_0 A \left[ 3 \left( L_{tot} - 2\Delta \right)^2 - 4A^2 \right]}{12 E B H^3} \right]
\]

This means that the shear deflection \(V_s\) has a constant value on this interval which also ought to be the summation of the expansion in cosines or sines terms. A proof will be given for the expansion in cosines terms while this expansion converge the best.

**“Shear Strain”**

In case of the deflection due to pure bending the related bending (horizontal) strain is derived from a 2\(^{nd}\) derivation of the deflection with respect to the distance \(X\). In fact this 2\(^{nd}\) derivation will give the moment \(M\) (see Part I of these series). In the static case the deflection polynomial between \(X = A + \Delta\) and \(X = L_{tot} - A - \Delta\) is of the order two (parabolic). Therefore the (bending) strain in this region will be a constant. The CEN formulas are based on this relationship by assuming the same parabolic equation for the amplitude of the deflection in case of a dynamic situation. The mass inertia forces due to the weight of the beam and other extra (moving) masses will disturb this simple relationship. Assuming that for single sine or cosine terms in the series expansions for the deflection due
to pure bending the 2\textsuperscript{nd} derivation of the deflection will lead to the correct strain value for this deflection component, the bending strains are calculated in the Excel program. As shown by calculations the differences in strain values with the static case are rather small for low frequencies and low masses for the beam. Except for the case in which the chosen location is equal or near to the position of the clamps (supports).

As can be obtained from the differential equation for the deflection $V_s$ due to shear, a (theoretical calculated) 2\textsuperscript{nd} derivation with respect to $x$ will lead to a zero value as expected, even in the case of dynamic loading (without extra moving masses). A shear force does not lead to a horizontal strain in the beam but will only deform the cross section of the beam. This deduction is based on the formulas for the static and dynamic case in which no extra moving masses are present. When an extra moving mass is present (e.g. the weight of the plunger) the two principle differential equations are coupled by the mass inertia force at the point where the mass is connected to the beam (often the two inner clamps or supports) and the reaction forces at the two outer supports. Due to the extra term the phase lag for the shear deflection will not have the same (but opposite) value as the phase lag of the complex stiffness modulus for the beam. The difference is small but exists. Therefore the “shear strain” “should” be calculated from the deflections which are based on an expansion in sines or cosines terms. However, especially for locations at and near the clamps will lead to unrealistic strain values. Performing the “required” 2\textsuperscript{nd} derivation will eliminate the convergence of the expansion (the factor $n^2$ will vanish in the denominator of the terms). Only the appearance of an extra moving mass may give some convergence.

**Shear Deflection $V_s$**

Omitting the term for the phase lag and the time function, the solution in a series development is given by equation II-6

$$V_s(x) = \sum_{n=1}^{\infty} \left[ H_{2n} \left( U_{2n}(x) - U_{2n}(\Delta) \right) \right] ; \quad H_{2n} = \frac{2F_0}{L_{\text{tot}}} \frac{2(1+\nu)}{\alpha B H} \left( \frac{L_{\text{tot}}^2}{(2n)^2\pi^2} \right)$$  \hspace{1cm} (II-6)

The series $U_n$ is given by equation II-7.

$$U_{2n}(x) = \cos \left( 2n\pi \frac{x}{L_{\text{tot}}} \right) \left[ \cos \left( 2n\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \cos \left( 2n\pi \frac{\Delta}{L_{\text{tot}}} \right) \right]$$  \hspace{1cm} (II-7)

Using equation II-4 equation II-7 can be rewritten as equation II-8

$$U_{2n}(x) = \cos \left( 2n\pi \frac{x}{L_{\text{tot}}} \right) \left[ \cos \left( 2n\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \cos \left( 2n\pi \frac{\Delta}{L_{\text{tot}}} \right) \right] =$$  \hspace{1cm} (II-8)

$$\cos \left( 2n\pi \frac{x-A-\Delta}{L_{\text{tot}}} \right) + \cos \left( 2n\pi \frac{x+A+\Delta}{L_{\text{tot}}} \right) - \cos \left( 2n\pi \frac{x-\Delta}{L_{\text{tot}}} \right) - \cos \left( 2n\pi \frac{x+\Delta}{L_{\text{tot}}} \right)$$
\[ U_{2n}(\Delta) = \cos\left(2n\pi \frac{\Delta}{L_{\text{tot}}} \right) \cdot \left[ \cos\left(2n\pi \frac{A+\Delta}{L_{\text{tot}}} \right) - \cos\left(2n\pi \frac{\Delta}{L_{\text{tot}}} \right) \right] = \]

\[ \frac{\cos\left(2n\pi \frac{A}{L_{\text{tot}}} \right) + \cos\left(2n\pi \frac{A+2\Delta}{L_{\text{tot}}} \right) - 1 - \cos\left(2n\pi \frac{2\Delta}{L_{\text{tot}}} \right)}{2} \]

If \( \Delta \) equals 0 equations II-8 and II-9 can be rewritten as:

\[ U_{2n}(x) = \frac{\cos\left(2n\pi \frac{x-A}{L_{\text{tot}}} \right) + \cos\left(2n\pi \frac{x+A}{L_{\text{tot}}} \right) - 2\cos\left(2n\pi \frac{x}{L_{\text{tot}}} \right)}{2}; \quad U_{2n}(0) = \left[ \cos\left(2n\pi \frac{A}{L_{\text{tot}}} \right) - 1 \right] \]

Due to the term \( n^2 \) in the denominator (see eq. II-6) these series can be summed up. Applying equation II-1 leads for the interval \( A+\Delta \leq x \leq L_{\text{tot}}/2 \) finally to the solution as given by equation 45:

\[ V_s\{A+\Delta \leq x \leq \frac{L_{\text{tot}}}{2}\} = \frac{(1+\mu)F_0^c}{\frac{S_{\text{mix}}^c}{\alpha B H}} \]

As mentioned before using the \( U_{2n}[x] \) series will lead to a reasonable fast convergence for the development of the series. This is due to the fact that the \( U_{2n}[x] \) series already satisfy one boundary restriction of the differential equation (shear force = 0 for \( x = 0 \)). Using the \( T_{2n-1}[x] \) series is a different ‘story’. Specially when \( \Delta \rightarrow 0 \) the convergence is ‘gone’.

**Horizontal “Shear Strain” Contribution(?)**

The title of this paragraph is somehow misleading. As can be seen by the equations for the shear deflection in chapter 7, the shear force will not contribute to the horizontal strain in the beam in the case that there are no extra moving masses. But even when extra moving masses are present, there ought to be no contribution. However, in that case the strain is calculated by taking the second derivate of the deflection. And hence the convergence of the series expansion is ‘lost’ while the term \((2n-1)^2\) in the denominator of the terms is eliminated by the derivation. Therefore it is better to calculate the strain in another way:

1. Define, in spite of the results of the 2nd derivate of the shear deflection, the contribution of the shear force to the horizontal strain as zero.

2. Multiply the calculated or measured total deflection according to the following formula: \( V_b'(x) = V_t(x) \cdot \frac{1}{\text{Ratio}(x) + 1} \)
3. For measuring the total deflection in the centre (L\text{tot}/2) the Ratio is equal to:

\[ \text{Ratio}(x = \frac{L_{\text{tot}}}{2}) = \frac{V_{x}(x = \frac{L_{\text{tot}}}{2})}{V_{b}(x = \frac{L_{\text{tot}}}{2})} = \frac{4(1 + \mu)H^{2}}{\alpha(3L_{\text{tot}}^{2} - 4A^{3})} \quad \text{[Pseudo-static case]} \]

And if the deflection is calculated or measured at the inner clamp by:

\[ \text{Ratio}(x = A + \Delta) = \frac{V_{x}(x = A)}{V_{b}(x = A)} = \frac{4(1 + \mu)H^{2}}{\alpha(3L_{\text{tot}}^{2} - 4A^{3})} \quad \text{[Pseudo-static case]} \]

4. In this way a good estimate for the deflection due to bending is obtained. By multiplying this deflection with the following factor a good estimate is found for the real horizontal strain value.

5. \[
R^{*}(x) = \frac{12(L_{\text{tot}} - 2\Delta)^{3}}{A} \cdot \frac{1}{3x(L_{\text{tot}} - 2\Delta) - 3x^{2} - A^{2}}
\]

\[
\varepsilon(x) = \frac{HA}{4(L_{\text{tot}} - 2\Delta)^{3}} \cdot R^{*}(x) \cdot V_{b}^{*}(x) = \frac{3H}{(3x(L_{\text{tot}} - 2\Delta) - 3x^{2} - A^{2})} \cdot V_{b}^{*}(x)
\]